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\mathcal{D} and $\frac{1}{\mathcal{D}}$ Operator

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1 Introduction

Mathematical equations approaching the true system are becoming more complex and demanding such that it is required some trivial formulas to deal with them. The most enigmatic inquiries in this discipline are differentiation and partial integral and those with more than two functions could be solved with these operators. This book covers the solutions of various differential and integral equations (ordinary and partial) indicating the physical problems, simultaneous, linear and non-homogeneous

2 \mathcal{D} and $\frac{1}{\mathcal{D}}$ - Operator

Definition. Operator \mathcal{D} is a differential operator being used to obtain the function derivative with respect to either x or t , such that

$$\mathcal{D}(y) := \frac{d}{dx}(y) = \frac{d}{dt}(y) \longrightarrow \mathcal{D}^n(y) := \frac{d^n}{dx^n}(y) = \frac{d^n}{dt^n}(y) \quad (1)$$

where $\mathcal{D}(x^5)$ and $\mathcal{D}^2(2x^3)$ would yield $5x^4$ and $12x$ in turn. This then could be further modified as $\mathcal{D} - 5$ and $5\mathcal{D}^2 - 2\mathcal{D} + 7$ to operate e^{7x} and $3x^3$ which generates $2e^{7x}$ and $90x - 18x^2 + 21x^3$ respectively, or even the question associated with $\mathcal{D}^2(\cos 13x)$ and $[\mathcal{D}^2 + 3\mathcal{D} - 8](\sin 4x)$. The theoretical ideas to deal with complex differential problems is interestingly discussed along with related questions.

Definition. Compared to the preceding, operator $\frac{1}{\mathcal{D}}$ is an integral operator with respect to certain independent variables, such that

$$\frac{1}{\mathcal{D}}(y) := \int y dx \longrightarrow \left[\frac{1}{\mathcal{D}}\right]^n(y) := \int^{(n)} y dt \quad (2)$$

for instances:

$$1) \frac{1}{\mathcal{D}^2}(4x^4) = \frac{1}{\mathcal{D}}\left(\frac{4}{5}x^5\right) = \frac{4}{30}x^6 + c \quad 2) \frac{1}{\mathcal{D}^2} \sin 4x = \frac{1}{\mathcal{D}}\left(-\frac{1}{4} \cos 4x\right) = -\frac{1}{16} \sin 4x + c$$

and the more complex example

$$\begin{aligned} 3) \left[\frac{1}{\mathcal{D}^2} + \frac{1}{\mathcal{D}} - 2\right](2x^4) &= \frac{1}{\mathcal{D}^2}(2x^4) + \frac{1}{\mathcal{D}}(2x^4) - 2(2x^4) \\ &= \frac{1}{\mathcal{D}}\left(\frac{2}{5}x^5\right) + \frac{2}{5}x^5 - 4x^4 = \frac{2}{30}x^6 + \frac{2}{5}x^5 - 4x^4 + c \end{aligned}$$

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Theorem 2.1. Suppose a function of $e^{\delta x}$ to be solved, this means that $\Psi[\mathcal{D}](e^{\delta x})$ equals to $\Psi[\delta]e^{\delta x}$. The result is just to complete the equation of \mathcal{D} depending on the value and the sign of δ . For the positive (+) and negative (-), those generate $+\delta \rightarrow +\mathcal{D}$ and $-\delta \rightarrow -\mathcal{D}$ in turn.

Proof. If operator \mathcal{D} is to solve $e^{\delta x}$, then the variables of \mathcal{D} is altered to δ .

$$\begin{aligned}\Psi[\mathcal{D}] &= a\mathcal{D}^2 + b\mathcal{D} + c \\ \Psi[\mathcal{D}](e^{\delta x}) &= [a\mathcal{D}^2 + b\mathcal{D} + c](e^{\delta x}) = a\mathcal{D}^2(e^{\delta x}) + b\mathcal{D}(e^{\delta x}) + c(e^{\delta x}) \\ &= a\delta^2 e^{\delta x} + b\delta e^{\delta x} + ce^{\delta x} = [a\delta^2 + b\delta + c]e^{\delta x} \longrightarrow \Psi[\delta]e^{\delta x}\end{aligned}$$

■

Example 2.1. If $\Psi[\mathcal{D}] = 5\mathcal{D}^2 + 7\mathcal{D} + 10$, find $\Psi[\mathcal{D}](e^{2x})$!

$$\Psi[\mathcal{D}](e^{2x}) = [5\mathcal{D}^2 + 7\mathcal{D} + 10](e^{2x}) = [20 + 14 + 10]e^{2x} \longrightarrow 44e^{2x}$$

Example 2.2. If $\Psi[\mathcal{D}] = 8\mathcal{D}^3 - 7\mathcal{D}^2 + 3\mathcal{D} - 4$, find $\Psi[\mathcal{D}](e^x)$!

$$\Psi[\mathcal{D}](e^x) = [8\mathcal{D}^3 - 7\mathcal{D}^2 + 3\mathcal{D} - 4](e^x) = [8 - 7 + 3 - 4]e^x \longrightarrow 0$$

Example 2.3. If $\Psi[\mathcal{D}] = 3\mathcal{D}^2 + 2\mathcal{D} - 6$, find $\Psi[\mathcal{D}](e^{-\frac{1}{2}x})$!

$$\Psi[\mathcal{D}](e^{-\frac{1}{2}x}) = [3\mathcal{D}^2 + 2\mathcal{D} - 6](e^{-\frac{1}{2}x}) = \left[-\frac{3}{4} - 1 - 6\right]e^{-\frac{1}{2}x} \longrightarrow -\frac{31}{4}e^{-\frac{1}{2}x}$$

Theorem 2.2. Suppose functions of $\sin \delta x$ and $\cos \delta x$, the second derivative of them would be $\Psi[\mathcal{D}^2](\sin \delta x) = \Psi[-\delta^2]\sin \delta x$ and also $\Psi[\mathcal{D}^2](\cos \delta x) = \Psi[-\delta^2]\cos \delta x$. This indicates that any positive δ being operated by \mathcal{D}^2 is substituted to $-\delta^2$

Proof. The second derivative of $\sin \delta x$ is denoted as changing \mathcal{D}^2 to $-\delta$. In general, for any n -multiples derivative with n -even number starting from 2, it results in $\mathcal{D}^n \rightarrow -\delta^n$

$$\Psi[\mathcal{D}](\sin \delta x) = [\mathcal{D}^2](\sin \delta x) = [\mathcal{D}](\delta \cos \delta x) = -\delta^2 \sin \delta x \longrightarrow \Psi[-\delta^2]\sin \delta x$$

This is exactly the same as if it is for $\cos \delta x$

■

Example 2.4. If $\Psi[\mathcal{D}] = 3\mathcal{D}^2 - 2\mathcal{D} + 1$, find $\Psi[\mathcal{D}](\sin 4x)$!

$$\begin{aligned}\Psi[\mathcal{D}](\sin 4x) &= [3\mathcal{D}^2 - 2\mathcal{D} + 1](\sin 4x) = 3\mathcal{D}^2(\sin 4x) - 2\mathcal{D}(\sin 4x) + \sin 4x \\ &= -48 \sin 4x - 8 \cos 4x + \sin 4x = -47 \sin 4x - 8 \cos 4x\end{aligned}$$

Example 2.5. If $\Psi[\mathcal{D}] = \mathcal{D}^2 + 3$, find $\Psi^{-1}[\mathcal{D}](\sin 3x)$!

$$\left[\frac{1}{\mathcal{D}^2 + 3}\right](\sin 3x) = \left[\frac{1}{-9 + 3}\right]\sin 3x = -\frac{1}{6}\sin 3x$$

Example 2.6. If $\Psi[\mathcal{D}] = \mathcal{D}^2 + 4\mathcal{D} - 2$, find $\Psi^{-1}[\mathcal{D}](\sin 2x)$!

$$\begin{aligned}\left[\frac{1}{\mathcal{D}^2 + 4\mathcal{D} - 2}\right](\sin 2x) &= \left[\frac{1}{-4 + 4\mathcal{D} - 2}\right](\sin 2x) = \left[\frac{1}{4\mathcal{D} - 6}\right](\sin 2x) \\ &= \left[\frac{1}{4\mathcal{D} - 6}\right]\left[\frac{4\mathcal{D} + 6}{4\mathcal{D} + 6}\right](\sin 2x) = \frac{4\mathcal{D} + 6}{16\mathcal{D}^2 - 36}(\sin 2x) \\ &= -\frac{1}{100}[4\mathcal{D} + 6](\sin 2x) \\ &= -\frac{1}{100}[8 \cos(2x) + 6 \sin(2x)]\end{aligned}$$

Theorem 2.3. Suppose a function of $e^{\delta x}\Phi(x)$ to be solved, this means that $\Psi[\mathcal{D}](e^{\delta x}\Phi(x))$ equals to $e^{\delta x}\Psi[\mathcal{D} + \delta]\Phi(x)$. The term $e^{\delta x}$ could be taken out from the calculation but the term of $\Psi[\mathcal{D}]$ should be changed by adding each term of \mathcal{D} with some constant variable of δ and leaving $\Phi(x)$ to be proceed.

Proof. The term of \mathcal{D} should be altered by arbitrary constant δ

$$\begin{aligned}\Psi[\mathcal{D}](e^{\delta x}\Phi(x)) &= [\mathcal{D}](e^{\delta x})\Phi(x) + e^{\delta x}[\mathcal{D}]\Phi(x) = \delta e^{\delta x}\Phi(x) + e^{\delta x}[\mathcal{D}]\Phi(x) \\ &= e^{\delta x}(\delta\Phi(x) + [\mathcal{D}]\Phi(x)) = e^{\delta x}\Psi[\mathcal{D} + \delta]\Phi(x)\end{aligned}$$

Example 2.7. If $\Psi[\mathcal{D}] = 2\mathcal{D} + 1$, find $\Psi[\mathcal{D}](e^x x)!$

$$\Psi[\mathcal{D}](e^x x) = [2\mathcal{D} + 1](e^x x) = e^x [2\mathcal{D} + 3]x \longrightarrow e^x(2 + 3x)$$

Example 2.8. If $\Psi[\mathcal{D}] = \frac{1}{\mathcal{D} + 2}$, find $\Psi[\mathcal{D}](e^{-2x})!$

$$\begin{aligned}\Psi[\mathcal{D}](e^{-2x}) &= \left[\frac{1}{\mathcal{D} + 2}\right](e^{-2x}) = \frac{1}{0}(e^{-2x}) \quad (\text{failed}) \\ &= \left[\frac{1}{\mathcal{D} + 2}\right](e^{-2x} \cdot 1) = e^{-2x} \left[\frac{1}{\mathcal{D}}\right](1) = xe^{-2x}\end{aligned}$$

Example 2.9. If $\Psi[\mathcal{D}] = \frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 3}$, find $\Psi[\mathcal{D}](e^x \sin 2x)!$

$$\begin{aligned}\Psi[\mathcal{D}](e^x \sin 2x) &= \left[\frac{1}{\mathcal{D}^2 - 2\mathcal{D} + 3}\right](e^x \sin 2x) = e^x \left[\frac{1}{(\mathcal{D} + 1)^2 - 2(\mathcal{D} + 1) + 3}\right](\sin 2x) \\ &= e^x \left[\frac{1}{\mathcal{D}^2 + 2}\right](\sin 2x) = e^x \left[\frac{1}{-4 + 2}\right](\sin 2x) \longrightarrow -\frac{1}{2}e^x(\sin 2x)\end{aligned}$$

Theorem 2.4. Suppose a function of $x\Phi(x)$ to be solved, this means that

$$\frac{1}{\mathcal{D}}(x\Phi(x)) = \left[x - \frac{1}{\mathcal{D}}\right] \frac{1}{\mathcal{D}}\Phi(x)$$

Proof. Using $\mathcal{D}(uv) = u'v + uv'$ and suppose $v' = \Phi(x)$ and $v = \frac{1}{\mathcal{D}}\Phi(x)$, it is then

$$\begin{aligned}\mathcal{D}\left(x\frac{1}{\mathcal{D}}\Phi(x)\right) &= \frac{1}{\mathcal{D}}\Phi(x) + x\Phi(x) \longrightarrow x\Phi(x) = \mathcal{D}\left(x\frac{1}{\mathcal{D}}\Phi(x)\right) - \frac{1}{\mathcal{D}}\Phi(x) \\ \frac{1}{\mathcal{D}}(x\Phi(x)) &= x\frac{1}{\mathcal{D}}\Phi(x) - \frac{1}{\mathcal{D}}\Phi(x) \longrightarrow \frac{1}{\mathcal{D}}(x\Phi(x)) = \left[x - \frac{1}{\mathcal{D}}\right] \frac{1}{\mathcal{D}}\Phi(x)\end{aligned}$$

Example 2.10. If $\Psi[\mathcal{D}] = \frac{1}{\mathcal{D}}$, find $\Psi[\mathcal{D}](x \sin x)!$

$$\frac{1}{\mathcal{D}}(x \sin x) = \left[x - \frac{1}{\mathcal{D}}\right] \frac{1}{\mathcal{D}} \sin x = \left[x - \frac{1}{\mathcal{D}}\right] - \cos x = -x \cos x + \sin x$$

Theorem 2.5. There are two ways of making integrals into series (derivative), using MacLaurin series and conventional derivative numerator over denominator. Keep in mind that the series is depending on the highest order n of derivative term, for instances

$$\begin{aligned}\frac{1}{\mathcal{D} + 1} &= 1 - \mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3 + \dots \\ \frac{1}{4\mathcal{D}^2 + \mathcal{D} + 2} &= \frac{1}{2} - \frac{1}{4}\mathcal{D} - \frac{7}{8}\mathcal{D}^2 + \dots \\ \frac{1}{\mathcal{D}^3 + 2\mathcal{D}^2 - \mathcal{D} + 5} &= \frac{1}{5} + \frac{1}{25}\mathcal{D} - \frac{9}{125}\mathcal{D}^2 + \dots \\ \frac{1}{\mathcal{D}^2 + 2\mathcal{D}} &= \frac{1}{\mathcal{D}} \frac{1}{\mathcal{D} + 2} = \frac{1}{\mathcal{D}} \left[\frac{1}{2} - \frac{1}{4}\mathcal{D} + \frac{1}{8}\mathcal{D}^2 - \dots\right]\end{aligned}$$

3 Integration with $\frac{1}{\mathcal{D}}$ Operator

To solve integrals with more than two integrand with different term is usually done by partial integrals. Moreover, If the order of each integrand is higher, then the solution should be finished in complex steps and repetitive. To deal with such laborious things, the usage of $\frac{1}{\mathcal{D}}$ operator could ease the calculation, applying the fifth theorems discussed in the preceding section. To understand comprehensively, it is better to jump into the given questions as follows.

Example 3.1. Answer this question $\int e^x \cos 2x dx$! Hint: Theorem (2.3)

$$\begin{aligned} \int e^x \cos 2x dx &= \left[\frac{1}{\mathcal{D}} \right] (e^x \cos 2x) = e^x \left[\frac{1}{\mathcal{D}+1} \right] \cos 2x & (\delta = 1) \\ &= e^x \left[\frac{\mathcal{D}-1}{\mathcal{D}^2-1} \right] \cos 2x = e^x \left[\frac{\mathcal{D}-1}{-5} \right] \cos 2x & (\text{Theorem (2.2)}) \\ &= -\frac{1}{5} e^x [\mathcal{D}-1] \cos 2x = \frac{1}{5} e^x [2 \sin 2x + \cos 2x] \end{aligned}$$

Example 3.2. Answer this question $\int x e^x \sin 3x dx$! Hint: Theorem (2.4)

$$\begin{aligned} \int x e^x \sin 3x dx &= \left[\frac{1}{\mathcal{D}} \right] (x e^x \sin 3x) = \left[x - \frac{1}{\mathcal{D}} \right] \left[\frac{1}{\mathcal{D}} \right] e^x \sin 3x & (\text{Theorem (2.3)}) \\ &= \left[x - \frac{1}{\mathcal{D}} \right] e^x \left[\frac{1}{\mathcal{D}+1} \right] \sin 3x = \left[x - \frac{1}{\mathcal{D}} \right] e^x \left[\frac{\mathcal{D}-1}{\mathcal{D}^2-1} \right] \sin 3x \\ &= \left[x - \frac{1}{\mathcal{D}} \right] e^x \left[\frac{\mathcal{D}-1}{-10} \right] \sin 3x = \left[x - \frac{1}{\mathcal{D}} \right] - \frac{1}{10} e^x [\mathcal{D}-1] \sin 3x \\ &= \left[x - \frac{1}{\mathcal{D}} \right] - \frac{1}{10} e^x [3 \cos 3x - \sin 3x] \\ &= \frac{1}{10} x e^x [\sin 3x - 3 \cos 3x] + \underbrace{\left[\frac{1}{\mathcal{D}} \right] \left(\frac{3}{10} e^x \cos 3x \right)}_A - \underbrace{\left[\frac{1}{\mathcal{D}} \right] \left(\frac{1}{10} e^x \sin 3x \right)}_B \\ A &= \frac{3}{10} e^x \left[\frac{1}{\mathcal{D}+1} \right] \cos 3x = \frac{3}{10} e^x \left[\frac{\mathcal{D}-1}{\mathcal{D}^2-1} \right] \cos 3x = \frac{3}{10} e^x \left[\frac{\mathcal{D}-1}{-10} \right] \cos 3x \\ &= -\frac{3}{100} e^x [\mathcal{D}-1] \cos 3x = -\frac{3}{100} e^x [-3 \sin 3x - \cos 3x] \\ B &= \frac{1}{10} e^x \left[\frac{1}{\mathcal{D}+1} \right] \sin 3x = \frac{1}{10} e^x \left[\frac{\mathcal{D}-1}{\mathcal{D}^2-1} \right] \sin 3x = \frac{1}{10} e^x \left[\frac{\mathcal{D}-1}{-10} \right] \sin 3x \\ &= -\frac{1}{100} e^x [\mathcal{D}-1] \sin 3x = -\frac{1}{100} e^x [3 \cos 3x - \sin 3x] \\ A - B &= \frac{1}{100} e^x [9 \sin 3x + 3 \cos 3x + 3 \cos 3x - \sin 3x] \\ \int x e^x \sin 3x dx &= \frac{1}{10} x e^x [\sin 3x - 3 \cos 3x] + \frac{1}{100} e^x [8 \sin 3x + 6 \cos 3x] \end{aligned}$$

Example 3.3. Answer this question $\int \ln x dx$! Hint: Theorem (2.3)

Suppose that $\ln x = u$, so that $e^u = x$ and $dx = e^u du$

$$\begin{aligned} \int \ln x dx &= \int u e^u du = \left[\frac{1}{\mathcal{D}} \right] (e^u u) = e^u \left[\frac{1}{\mathcal{D}+1} \right] u \text{ (or) } \left[u - \frac{1}{\mathcal{D}} \right] \frac{1}{\mathcal{D}} e^u & (\text{Theorem (2.4)}) \\ &= e^u [1 - \mathcal{D}] u = e^u [u - 1] = x (\ln x - 1) \end{aligned}$$

Example 3.4. Answer this question $\int \sin(\ln x) dx$! Hint: Theorem (2.3)

Suppose that $\ln x = u$, so that $e^u = x$ and $dx = e^u du$

$$\begin{aligned}\int \sin(\ln x) dx &= \int \sin(u) e^u du = \left[\frac{1}{\mathcal{D}} \right] (e^u \sin u) = e^u \left[\frac{1}{\mathcal{D} + 1} \right] \sin u \\ &= e^u \left[\frac{\mathcal{D} - 1}{\mathcal{D}^2 - 1} \right] \sin u = e^u \left[\frac{\mathcal{D} - 1}{-2} \right] \sin u = -\frac{1}{2} e^u [\mathcal{D} - 1] \sin u \\ &= -\frac{1}{2} e^u [\cos u - \sin u] = -\frac{1}{2} x [\cos(\ln x) - \sin(\ln x)]\end{aligned}$$

Example 3.5. Answer this question $\int \frac{\sin(\ln x)}{x^3} dx$! Hint: Theorem (2.3)

Suppose that $\ln x = u$, so that $e^u = x$ and $dx = e^u du$

$$\begin{aligned}\int \frac{\sin(\ln x)}{x^3} dx &= \int \frac{\sin(u)}{e^{3u}} e^u du = \left[\frac{1}{\mathcal{D}} \right] (e^{-2u} \sin u) = e^{-2u} \left[\frac{1}{\mathcal{D} - 2} \right] \sin u \\ &= e^{-2u} \left[\frac{\mathcal{D} + 2}{\mathcal{D}^2 - 4} \right] \sin u = e^{-2u} \left[\frac{\mathcal{D} + 2}{-5} \right] \sin u = -\frac{1}{5} e^{-2u} [\mathcal{D} + 2] \sin u \\ &= -\frac{1}{5} e^{-2u} [\cos u + 2 \sin u] = -\frac{1}{5} x^{-2} [\cos(\ln x) + 2 \sin(\ln x)]\end{aligned}$$

Example 3.6. Answer this question $\int \frac{\ln x}{x^4} dx$! Hint: Theorem (2.3)

Suppose that $\ln x = u$, so that $e^u = x$ and $dx = e^u du$

$$\begin{aligned}\int \frac{\ln x}{x^4} dx &= \int \frac{u}{e^{4u}} e^u du = \left[\frac{1}{\mathcal{D}} \right] (e^{-3u} u) = \left[u - \frac{1}{\mathcal{D}} \right] \frac{1}{\mathcal{D}} e^{-3u} \\ &= \left[u - \frac{1}{\mathcal{D}} \right] - \frac{1}{3} e^{-3u} = -\frac{1}{3} u e^{-3u} + \left[\frac{1}{\mathcal{D}} \right] \frac{1}{3} e^{-3u} \\ &= -\frac{1}{3} u e^{-3u} - \frac{1}{9} e^{-3u} = -\frac{1}{3x^3} (\ln x + 1)\end{aligned}$$

Exercise 1. Solve the following integral equations!

1. $\int e^{2x} \cos 3x \cos 2x dx$ 2. $\int x e^x \cos x dx$ 3. $\int x e^x \sin^2 x dx$	4. $\int \frac{\ln x}{x^4} dx$ 5. $\int x e^x \cos^2 x dx$ 6. $\int \frac{\cos(\ln x)}{x^5} dx$
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The theory of Gottfried Wilhelm Leibniz known as "Leibniz's rule" is proposed as an alternative of Theorem (2.5). This is the equation with order n -th means the n -th derivative of the product (uv) , therefore

$$\begin{aligned}\mathcal{D}^n(uv) &= u \mathcal{D}^n(v) + n \mathcal{D}(u) \mathcal{D}^{n-1}(v) + \frac{1}{2!} n(n-1) \mathcal{D}^2(u) \mathcal{D}^{n-2}(v) + \\ &\quad \frac{1}{3!} n(n-1)(n-2) \mathcal{D}^3(u) \mathcal{D}^{n-3}(v) + \dots\end{aligned}\tag{3}$$

by changing the constant n with -1 , it results in the following function which is used to solve integral equations and also as an alternative of Theorem (2.5)

$$\frac{1}{\mathcal{D}}(uv) = u \frac{1}{\mathcal{D}}(v) - \mathcal{D}(u) \frac{1}{\mathcal{D}^2}(v) + \mathcal{D}^2(u) \frac{1}{\mathcal{D}^3}(v) - \mathcal{D}^3(u) \frac{1}{\mathcal{D}^4}(v) + \dots\tag{4}$$

Example 3.7. Answer this question $\int x^3 e^x dx$! Suppose that $u = x^3$ and $v = e^x$

$$\mathcal{D}(x^3) = 3x^2 \longrightarrow \mathcal{D}^2(x^3) = 6x \longrightarrow \mathcal{D}^3(x^3) = 6 \longrightarrow \mathcal{D}^4(x^3) = 0 \quad (5)$$

$$\left[\frac{1}{\mathcal{D}}\right] e^x = e^x \longrightarrow \left[\frac{1}{\mathcal{D}^2}\right] e^x = e^x \longrightarrow \left[\frac{1}{\mathcal{D}^3}\right] e^x = e^x \longrightarrow \left[\frac{1}{\mathcal{D}^4}\right] e^x = e^x \quad (6)$$

such that by substituting the Eqs.(5) and (6), the final solution is

$$\int x^3 e^x dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x = (x^3 - 3x^2 + 6x - 6) e^x + c$$

Example 3.8. Answer this question $\int x^3 \cos(ax) dx$! Suppose that $u = x^3$ and $v = \cos(ax)$

$$\mathcal{D}(x^3) = 3x^2 \longrightarrow \mathcal{D}^2(x^3) = 6x \longrightarrow \mathcal{D}^3(x^3) = 6 \longrightarrow \mathcal{D}^4(x^3) = 0 \quad (7)$$

$$\begin{aligned} \left[\frac{1}{\mathcal{D}}\right] \cos(ax) &= \frac{1}{a} \sin(ax) \longrightarrow \left[\frac{1}{\mathcal{D}^2}\right] \cos(ax) = -\frac{1}{a^2} \cos(ax) \longrightarrow \dots \\ \dots \left[\frac{1}{\mathcal{D}^3}\right] \cos(ax) &= -\frac{1}{a^3} \sin(ax) \longrightarrow \left[\frac{1}{\mathcal{D}^4}\right] e^x = \frac{1}{a^4} \cos(ax) \end{aligned} \quad (8)$$

such that by substituting the Eqs.(7) and (8), the final solution is

$$\begin{aligned} \int x^3 \cos ax dx &= x^3 \frac{1}{a} \sin(ax) + 3x^2 \frac{1}{a^2} \cos(ax) - 6x \frac{1}{a^3} \sin(ax) - 6 \frac{1}{a^4} \cos(ax) \\ &= \left(\frac{x^3}{a} - \frac{6x}{a^3}\right) \sin ax + \left(\frac{3x^2}{a^2} - \frac{6}{a^4}\right) \cos ax + c \end{aligned}$$

Example 3.9. Answer this question $\int x^3 (\ln x)^2 dx$! Hint: $\ln x = u \rightarrow e^u = x$ and $dx = e^u du$

$$\int x^3 (\ln x)^2 dx = \int e^{3u} u^2 e^u dx = \int e^{4u} u^2 dx = \left[\frac{1}{\mathcal{D}}\right] u^2 e^{4u}$$

where $u = \ln x$ and $v = e^{4u}$, therefore applying the Leibniz theory, it results in

$$\mathcal{D}(u^2) = 2u \longrightarrow \mathcal{D}^2(u^2) = 2 \longrightarrow \mathcal{D}^3(u^2) = 0 \quad (9)$$

$$\left[\frac{1}{\mathcal{D}}\right] e^{4u} = \frac{1}{4} e^{4u} \longrightarrow \left[\frac{1}{\mathcal{D}}\right] e^{4u} = \frac{1}{16} e^{4u} \longrightarrow \left[\frac{1}{\mathcal{D}}\right] e^{4u} = \frac{1}{64} e^{4u} \quad (10)$$

such that by substituting the Eqs.(9) and (10), the final solution is

$$\int x^3 (\ln x)^2 dx = u^2 \frac{1}{4} e^{4u} - 2u \frac{1}{16} e^{4u} + 2 \frac{1}{64} e^{4u} = \left(\frac{u^2}{4} - \frac{u}{8} + \frac{1}{32}\right) e^{4u}$$

where $u = \ln x$, by re-substituting, it yields

$$= \left[\frac{(\ln x)^2}{4} - \frac{\ln x}{8} + \frac{1}{32}\right] x^4 + c$$

Exercise 2. Solve the following integral equations!

$$1. \int x^3 e^{2x} dx \quad \Bigg| \quad 2. \int x^3 \sin(\ln x) dx$$

4 Solution of Ordinary Differential Equations

4.1 Ordinary Method

Further benefits of operator-D is to solve differential equations, especially second-order differential equations with constant coefficients. Suppose the general form of the second-order, (y) as dependant with respect to independent (x) variable along some constant variables (a, b, c) , to be solved is given as follows,

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = \Phi(x) \quad (11)$$

then the solution (y) of Eq.(11) is represented as the addition of complementary or the homogeneous solution (y_c) and particular solution (y_p) , such that

$$y = y_c + y_p \quad (12)$$

To solve complementary solution, the right-hand side of the given function in Eq.(11) is supposed to be zero and the counterpart left-side is constructed in terms of \mathcal{D} -term, therefore

$$a\mathcal{D}^2 + b\mathcal{D} + c = 0 \quad \longrightarrow \quad \mathcal{D}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (13)$$

The complementary solution is formed as exponential (e) term along with some constants (k_1, k_2) as given in the question. However, from the characteristic equation as Eq.(13), it can be seen that there are three different possibilities of roots meaning that there should be also the same number of formulating the complementary solution. They could be:

1) different $(\mathcal{D}_1 \neq \mathcal{D}_2)$ as Eq.(14a),

$$y_c = k_1 e^{\mathcal{D}_1 x} + k_2 e^{\mathcal{D}_2 x} \quad \longrightarrow \quad k_1 e^{\mathcal{D}_1 x} + k_2 e^{\mathcal{D}_2 x} + \dots + k_n e^{\mathcal{D}_n x}, \forall n > 2 \quad (14a)$$

2) the same $(\mathcal{D}_1 = \mathcal{D}_2)$ as Eq.(14b),

$$y_c = (k_1 + k_2 x) e^{\mathcal{D} x} \quad \longrightarrow \quad (k_1 + k_2 x + \dots + k_n x^{n-1}) e^{\mathcal{D} x}, \forall n > 2 \quad (14b)$$

3) imaginary roots with $(\mathcal{D}_1 = p + iq)$ and $(\mathcal{D}_2 = p - iq)$ as Eq.(14c), such that

$$y_c = k_1 e^{(p+iq)x} + k_2 e^{(p-iq)x} = (k_1 e^{qix} + k_2 e^{-qix}) e^{px} \quad (14c)$$

or since e^{qix} and e^{-qix} equal to $\cos(qx)$ and $\sin(qx)$ in turn, then Eq.(14d) could be written as

$$y_c = (k_1 \cos qx + k_2 \sin qx) e^{px} = k e^{px} \cos (qx \pm \theta) \quad (14d)$$

To solve particular solution, in terms of \mathcal{D} , the given question as Eq.(11) is computed in a whole, without assumption, with the insight from preceding properties, Theorem (2.1) – (2.5), such that

$$y_p = \frac{1}{a\mathcal{D}^2 + b\mathcal{D} + c} \Phi(x) \quad (15)$$

Example 4.1. Find the solution of the given question $\frac{d^3 y}{dx^3} - 2 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 8y = 0$. That is then written as $[\mathcal{D}^3 - 2\mathcal{D}^2 + 4\mathcal{D} - 8] y = 0$ and the solutions are,

1) Complementary y_c :

$$\begin{aligned} [\mathcal{D}^3 - 2\mathcal{D} + 4\mathcal{D} - 8] &= 0 \longrightarrow [\mathcal{D} - 2] [\mathcal{D}^2 + 4] = 0 \\ \mathcal{D}_1 = 2, \mathcal{D}_2 = 2i, \mathcal{D}_3 = -2i, \text{ such that } y_c &= k_1 e^{2x} + \underbrace{k_2 e^{2ix} + k_3 e^{-2ix}}_{k_2 \cos(2x) + k_3 \sin(2x)} \end{aligned}$$

2) Particular y_p : Since the right-hand side equals zero, then $y_p = 0$ therefore

$$\begin{aligned} y &= y_c = k_1 e^{2x} + k_2 e^{2ix} + k_3 e^{-2ix} \\ &= k_1 e^{2x} + k_2 \cos(2x) + k_3 \sin(2x) = k_1 e^{2x} + k \cos(2x - \theta) \end{aligned}$$

Example 4.2. The system is written as differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$. It could be also denoted as $[\mathcal{D}^2 - 3\mathcal{D} + 2]y = 0$ and the solution are

1) Complementary y_c :

$$[\mathcal{D}^2 - 3\mathcal{D} + 2] = 0 \longrightarrow \mathcal{D}_1 = 2, \mathcal{D}_2 = 1 \text{ such that } y_c = k_1 e^{2x} + k_2 e^x$$

2) Particular y_p : Since the right-hand side equals zero, then $y_p = 0$ and $y = y_c$

Example 4.3. Given the differential equation, $\frac{dy}{dt} + 2y = 1 + 2t + e^{-t}$ with $y(0) = 0$. It could be formed in terms of \mathcal{D} , resulting $[\mathcal{D} + 2]y = 1 + 2t + e^{-t}$ and the solutions are,

1) Complementary y_c :

$$[\mathcal{D} + 2] = 0 \longrightarrow \mathcal{D} = -2 \text{ such that } y_c = k e^{-2t}$$

2) Particular y_p :

$$\begin{aligned} y_p &= \left[\frac{1}{\mathcal{D} + 2} \right] (1 + 2t + e^{-t}) = \left[\frac{1}{\mathcal{D} + 2} \right] 1 + \left[\frac{1}{\mathcal{D} + 2} \right] 2t + \left[\frac{1}{\mathcal{D} + 2} \right] e^{-t} \\ &= \left[\frac{1}{2} - \frac{1}{4}\mathcal{D} \right] 1 + \left[\frac{1}{2} - \frac{1}{4}\mathcal{D} \right] 2t + e^{-t} \\ &= \frac{1}{2} - 0 + t - \frac{1}{2} + e^{-t} \longrightarrow t + e^{-t} \end{aligned}$$

such that the solution $y = y_c + y_p$ is $k e^{-2t} + t + e^{-t}$. Since the constraint of $y(0) = 0 \rightarrow k = -1$, then the final result would be $y = -e^{-2t} + t + e^{-t}$

Example 4.4. Solve the following differential equation $\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} = 0$ with $y(0) = y'(0) = y''(0) = 0$ and $y^{(3)}(0) = 1$. The respective \mathcal{D} -term is $[\mathcal{D}^4 + \mathcal{D}^2]y = 0$ and the solutions are

1) Complementary y_c :

$$[\mathcal{D}^4 + \mathcal{D}^2] = 0 \longrightarrow \mathcal{D}^2 [\mathcal{D}^2 + 1] = 0$$

$$\mathcal{D}_1 = \mathcal{D}_2 = 0, \mathcal{D}_3 = i, \mathcal{D}_4 = -i \text{ such that } y_c = (k_1 + k_2x) + k_3 e^x + k_4 e^{-x}$$

2) Particular y_p : Since the right-hand side equals zero, then $y_p = 0$ and $y = y_c$. Then, from the constraints, the final solution is presented as $y_c = (k_1 + k_2x) + k_3 \cos(x) + k_4 \sin(x)$,

$$\begin{aligned} y(x) &= (k_1 + k_2x) + k_3 \cos(x) + k_4 \sin(x) & y(0) &= 0 = k_1 + k_3 \\ y'(x) &= k_2 - k_3 \sin(x) + k_4 \cos(x) & y'(0) &= 0 = k_2 + k_4 \\ y''(x) &= -k_3 \cos(x) - k_4 \sin(x) & y''(0) &= 0 = -k_3 \\ y^{(3)}(x) &= k_3 \sin(x) - k_4 \cos(x) & y^{(3)}(0) &= 1 = k_3 - k_4 \end{aligned}$$

and it can be concluded that $k_1 = k_3 = 0$, $k_2 = 1$ and $k_4 = -1$, and it results in $y = x - \sin(x)$

Example 4.5. Supposed a differential equation representing a system, $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 5$. The \mathcal{D} transformation is then defined as $[\mathcal{D}^2 + 4\mathcal{D} + 3]y = 5$ and the solutions are,

1) Complementary y_c :

$$[\mathcal{D}^2 + 4\mathcal{D} + 3] = 0 \longrightarrow \mathcal{D}_1 = -1, \mathcal{D}_2 = -3 \text{ such that } y_c = k_1 e^{-x} + k_2 e^{-3x}$$

2) Particular y_p :

$$\begin{aligned} y_p &= \left[\frac{1}{\mathcal{D}^2 + 4\mathcal{D} + 3} \right] (5) = \left[\frac{1}{3} - \frac{4}{9}\mathcal{D} \right] (5) \\ &= \frac{5}{3} - 0 \longrightarrow \frac{5}{3} \end{aligned}$$

such that the final solution $y = y_c + y_p$ is $k_1 e^{-x} + k_2 e^{-3x} + \frac{5}{3}$, since there is no constraints

Example 4.6. The system representation is given as $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 10$. The modified \mathcal{D} -term is written as $[\mathcal{D}^2 + 2\mathcal{D}] y = 10$ and the solutions are,

1) Complementary y_c :

$$[\mathcal{D}^2 + 2\mathcal{D}] = 0 \longrightarrow \mathcal{D}_1 = 0, \mathcal{D}_2 = -2 \text{ such that } y_c = k_1 + k_2 e^{-2x}$$

2) Particular y_p :

$$\begin{aligned} y_p &= \left[\frac{1}{\mathcal{D}^2 + 2\mathcal{D}} \right] (10) = \left[\frac{1}{\mathcal{D} + 2} \right] \frac{1}{\mathcal{D}} (10) = \left[\frac{1}{\mathcal{D} + 2} \right] (10x) \\ &= \left[\frac{1}{2} - \frac{1}{4}\mathcal{D} \right] (10x) \longrightarrow 5x - \frac{5}{2} \end{aligned}$$

such that the final solution $y = y_c + y_p$ is $k_1 + k_2 e^{-2x} + 5x - \frac{5}{2}$ due to lack of further constraints

Example 4.7. Given the differential equation $\frac{d^2y}{dx^2} - y = 2^x$. The \mathcal{D} -term is further denoted as $[\mathcal{D}^2 - 1] y = 2^x$ and the solutions are,

1) Complementary y_c :

$$[\mathcal{D}^2 - 1] = 0 \longrightarrow \mathcal{D}_1 = 1, \mathcal{D}_2 = -1 \text{ such that } y_c = k_1 e^x + k_2 e^{-x}$$

2) Particular y_p :

$$y_p = \left[\frac{1}{\mathcal{D}^2 - 1} \right] (2^x) = \left[\frac{1}{\ln^2(2) - 1} \right] e^{x \ln(2)} \quad \text{where: } 2^x = e^{x \ln(2)}$$

such that the final solution $y = y_c + y_p$ is $k_1 e^x + k_2 e^{-x} + \frac{2^x}{\ln^2(2) - 1}$ due to the absence of constraints

Example 4.8. Find the solution of differential equation $\frac{d^2y}{dx^2} + 4y = \sin(2x)$. The \mathcal{D} transformation is then written as $[\mathcal{D}^2 + 4] y = \sin(2x)$ and the solutions are,

1) Complementary y_c :

$$[\mathcal{D}^2 + 4] = 0 \longrightarrow \mathcal{D}_1 = 2i, \mathcal{D}_2 = -2i \text{ such that } y_c = \underbrace{k_1 e^{2ix} + k_2 e^{-2ix}}_{k \cos(2x - \theta)}$$

2) Particular y_p : Note that $e^{\pm i\theta} = \cos(\theta) \pm i \sin(\theta)$ or $\cos(\theta) = \operatorname{Re}(e^{i\theta})$ and $\sin(\theta) = \operatorname{Im}(e^{i\theta})$

$$y_p = \left[\frac{1}{\mathcal{D}^2 + 4} \right] \sin(2x) = \left[\frac{1}{-2^2 + 4} \right] \sin(2x) = \frac{1}{0} \sin(2x) \quad (\text{failed})$$

$$= \operatorname{Im} \left[\frac{1}{\mathcal{D}^2 + 4} \right] e^{2ix} = \operatorname{Im} \left[\frac{1}{(2i)^2 + 4} \right] e^{2ix} = \operatorname{Im} \left[\frac{1}{0} e^{2ix} \right] \quad (\text{failed})$$

$$= \operatorname{Im} \left(\left[\frac{1}{\mathcal{D}^2 + 4} \right] e^{2ix} \cdot 1 \right) = \operatorname{Im} \left(e^{2ix} \left[\frac{1}{(\mathcal{D} + 2i)^2 + 4} \right] 1 \right) = \operatorname{Im} \left(e^{2ix} \left[\frac{1}{\mathcal{D} + 4i} \right] \frac{1}{\mathcal{D}} 1 \right)$$

$$= \operatorname{Im} \left(e^{2ix} \left[\frac{1}{\mathcal{D} + 4i} \right] x \right) = \operatorname{Im} \left(e^{2ix} \left[\frac{1}{4i} - \frac{1}{16i^2} \mathcal{D} \right] x \right) = \operatorname{Im} \left(e^{2ix} \left[\frac{x}{4i} + \frac{1}{16} \right] \right)$$

$$= \operatorname{Im} \left(e^{2ix} \left[\frac{x}{4i} + \frac{1}{16} \right] \right) = \operatorname{Im} \left([\cos(2x) + i \sin(2x)] \left[-\frac{ix}{4} + \frac{1}{16} \right] \right)$$

$$= \operatorname{Im} \left(-\frac{ix}{4} \cos(2x) + \frac{1}{16} \cos(2x) + \frac{x}{4} \sin(2x) + \frac{i}{16} \sin(2x) \right) \quad (2 \text{ and } 3\text{-term: } \operatorname{Re})$$

$$= \frac{1}{16} \sin(2x) - \frac{x}{4} \cos(2x)$$

such that the final solution $y = y_c + y_p$ is $k_1 e^{2ix} + k_2 e^{-2ix} - \frac{1}{4}x \cos(2x) + \frac{1}{16} \sin(2x)$ due to lack of further constraints

Example 4.9. Supposed a function $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^{2x} \cosh(x)$. The respective \mathcal{D} -term is figured out as $[\mathcal{D}^2 - 2\mathcal{D}] y = e^{2x} \cosh(x)$ and the solutions are,

1) Complementary y_c :

$$[\mathcal{D}^2 - 2\mathcal{D}] = 0 \longrightarrow \mathcal{D}_1 = 0, \mathcal{D}_2 = 2 \text{ such that } y_c = k_1 + k_2 e^{2x}$$

2) Particular y_p :

$$\begin{aligned} y_p &= \left[\frac{1}{\mathcal{D}^2 - 2\mathcal{D}} \right] e^{2x} \cosh(x) \quad \text{where: } \cosh(x) = \frac{e^x + e^{-x}}{2} \\ &= \frac{1}{2} \left[\frac{1}{\mathcal{D}^2 - 2\mathcal{D}} \right] (e^{3x} + e^x) = \frac{1}{2} \left[\frac{1}{3} e^{3x} - e^x \right] \longrightarrow \frac{1}{6} e^{3x} - \frac{1}{2} e^x \end{aligned}$$

such that the final solution $y = y_c + y_p$ is $k_1 + k_2 e^{2x} + \frac{1}{6} e^{3x} - \frac{1}{2} e^x$ due to the absence of constraints

Exercise 3. Solve the following differential equations!

1. $\frac{d^2y}{dx^2} - 16y = e^{4x}$
2. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = -4e^x \sinh(x)$ with $y(0) = 2$ and $y'(0) = 0$
3. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 8 \cos^2(x)$
4. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 34 \sin(x) \cos(x)$
5. $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = -e^{-t} [18 \cos(3t) + 26 \sin(3t)]$ with $y(0) = 0$ and $y'(0) = 6$

4.2 Parameter Variation Method I

Recalling Eq.(11), while the general solution comprises the same as $y = y_c + y_p$, yet each part of such solution could be modified, resulting divergent ways of finding the particular (y_p). Suppose the complementary $y_c = k_1 \psi_1 + k_2 \psi_2$, then the particular is denoted as

$$y_p = p\psi_1 + q\psi_2 \tag{16}$$

where

$$p = \int \frac{-\psi_2 \Phi(x)}{\psi_1 \psi_2' - \psi_1' \psi_2} dx \quad \text{and} \quad q = \int \frac{\psi_1 \Phi(x)}{\psi_1 \psi_2' - \psi_1' \psi_2} dx \tag{17}$$

Example 4.10. Solve the following differential equation, $\frac{d^2y}{dx^2} + y = \csc(x)$. The step is to find the \mathcal{D} -term followed by the complementary, which is preferred to the sine and cosine term instead of exponential, and particular solution, such that

1) Complementary y_c :

$$[\mathcal{D}^2 + 1] = 0 \longrightarrow \mathcal{D}_1 = i, \mathcal{D}_2 = -i \text{ such that } y_c = k_1 \cos(x) + k_2 \sin(x)$$

2) Particular y_p : $\psi_1 = \cos(x) \rightarrow \psi_1' = -\sin(x)$ and $\psi_2 = \sin(x) \rightarrow \psi_2' = \cos(x)$

$$\begin{aligned} p &= \int \frac{-\sin(x) \csc(x)}{\cos^2(x) + \sin^2(x)} dx = -x \quad \text{and} \quad q = \int \frac{\cos(x) \csc(x)}{\cos^2(x) + \sin^2(x)} dx = \ln(\varphi) \\ y_p &= -x \cos(x) + \sin(x) \ln(\varphi) \quad \text{with: } \varphi = \sin(x) \end{aligned}$$

This means the final solution $y = y_c + y_p$ is $k_1 \cos(x) + k_2 \sin(x) - x \cos(x) + \sin(x) \ln(\varphi)$ due to the absence of constraints

Example 4.11. Find the solution of such function, $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$!

1) Complementary y_c :

$$[\mathcal{D}^2 - 1] = 0 \longrightarrow \mathcal{D}_1 = 1, \mathcal{D}_2 = -1 \text{ such that } y_c = k_1 e^x + k_2 e^{-x}$$

2) Particular y_p : $\psi_1 = e^x \rightarrow \psi'_1 = e^x$ and $\psi_2 = e^{-x} \rightarrow \psi'_2 = -e^{-x}$

$$p = \int \frac{e^{-x}}{1+e^x} dx = \int \left[\frac{1}{e^x} - \frac{e^{-x}}{1+e^{-x}} \right] dx \quad \text{and} \quad q = - \int \frac{e^x}{1+e^x} dx$$

such that, $p = -e^{-x} + \ln(1+e^x)$ and $q = \ln(1+e^x)$. This means the final solution $y = y_c + y_p$ is $k_1 e^x + k_2 e^{-x} + [-e^{-x} + \ln(1+e^x)] e^x - e^{-x} \ln(1+e^x)$

Exercise 4. Solve the following differential equations!

$$1. \frac{d^2y}{dx^2} + y = \tan(x) \quad \left| \quad 2. \frac{d^2y}{dx^2} + y = \sec(x) \tan(x) \right.$$

4.3 Parameter Variation Method II

This section is specifically intended to solve particular solution, having the \mathcal{D} -term characteristic of the conjugate which could be simplified into the following method. If it is obtained that

$$\left[\frac{1}{\mathcal{D} - a} \right] \phi(x) = y \longleftrightarrow [\mathcal{D} - a] y = \phi(x), \text{ then the solution is } e^{ax} \int e^{-ax} \phi(x) dx \quad (18)$$

Example 4.12. Given the differential equation, $\frac{d^2y}{dx^2} + a^2y = \sec(ax)$, find the solution!

1) Complementary y_c :

$$[\mathcal{D}^2 + a^2] = 0 \longrightarrow \mathcal{D}_1 = ia, \mathcal{D}_2 = -ia \text{ such that } y_c = k_1 e^{ia} + k_2 e^{-ia}$$

2) Particular y_p : Note that $e^{\pm iax} = \cos(ax) \pm i \sin(ax)$

$$\begin{aligned} y_p &= \left[\frac{1}{\mathcal{D}^2 + a^2} \right] \sec(ax) \longrightarrow \frac{1}{\mathcal{D}^2 + a^2} = -\frac{1}{2ia} \frac{1}{\mathcal{D} + ia} + \frac{1}{2ia} \frac{1}{\mathcal{D} - ia} \\ &= -\frac{1}{2ia} \frac{1}{\mathcal{D} + ia} \sec(ax) + \frac{1}{2ia} \frac{1}{\mathcal{D} - ia} \sec(ax) \\ \frac{1}{\mathcal{D} \pm ia} \sec(ax) &= e^{\mp iax} \int e^{\pm iax} \sec(ax) dx = e^{\mp iax} \int \frac{\cos(ax) \pm i \sin(ax)}{\cos(ax)} dx \\ &= e^{\mp iax} \left[\int dx \pm i \int \tan(x) dx \right] = e^{\mp iax} \left[x \mp \frac{i}{a} \ln(\varphi) \right] \text{ with } \varphi = \cos(ax) \\ y_p &= -\frac{1}{2ia} \left(e^{-iax} \left[x - \frac{i}{a} \ln(\varphi) \right] \right) + \frac{1}{2ia} \left(e^{iax} \left[x + \frac{i}{a} \ln(\varphi) \right] \right) \\ &= x \left[\frac{e^{iax} - e^{-iax}}{2ia} \right] + \frac{1}{a} \ln(\varphi) \left[\frac{e^{iax} + e^{-iax}}{2a} \right] = x \sin(ax) + \frac{1}{a} \ln(\varphi) \cos(ax) \end{aligned}$$

From this, it is concluded that the final solution is $y = k_1 e^{ia} + k_2 e^{-ia} + x \sin(ax) + \frac{1}{a} \ln(\varphi) \cos(ax)$

Exercise 5. Solve the following differential equations!

$$1. \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 2y = e^{-x} \sec^3(x) \quad \left| \quad 2. \frac{d^2y}{dx^2} + 4y = \tan(2x) \right.$$

5 Solution of Linear Homogeneous Differential Equations

This section focuses on the homogeneous functions and the general form of the linear homogeneous differential equation is denoted as follows,

$$\varphi_k x^k \frac{d^k y}{dy^k} + \varphi_{k-1} x^{k-1} \frac{d^{k-1} y}{dy^{k-1}} + \varphi_{k-2} x^{k-2} \frac{d^{k-2} y}{dy^{k-2}} + \dots + \varphi_0 y = \Phi(x) \quad (19)$$

where $\varphi_0, \varphi_1, \varphi_2, \dots$ are the arbitrary constants. Supposed $x = e^z$, then $z = \ln(x)$, such that each term in the series with an assumption $\varphi_n = 1, \forall n = 0 : k$ is defined as a sequence of an infinite product

$$x^n \frac{d^n y}{dx^n} = \left(\prod_{n=1}^k [\mathcal{D} - (n-1)] \right) y = \mathcal{D} [\mathcal{D} - 1] [\mathcal{D} - 2] \dots [\mathcal{D} - (k-1)] y \quad (20)$$

and the detail explanations are enlightened in the following proofs, therefore

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \longrightarrow x \frac{dy}{dx} = \mathcal{D} y \quad \text{where: } \frac{d}{dz} = \mathcal{D} \quad (21)$$

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dz} \right) \\ &= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d^2 y}{dz^2} \frac{dz}{dx} = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x^2} \frac{d^2 y}{dz^2} \\ &= \frac{1}{x^2} [\mathcal{D}^2 - \mathcal{D}] y \longrightarrow x^2 \frac{d^2 y}{dx^2} = \mathcal{D} [\mathcal{D} - 1] y \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{d^3 y}{dx^3} &= \frac{d}{dx} \left(\frac{d}{dx} \frac{dy}{dx} \right) = \frac{d}{dx} \left(-\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x^2} \frac{d^2 y}{dz^2} \right) \\ &= \frac{2}{x^3} \frac{dy}{dz} - \frac{1}{x^2} \frac{d^2 y}{dz^2} \frac{dz}{dx} - \frac{2}{x^3} \frac{d^2 y}{dz^2} + \frac{1}{x^2} \frac{d^3 y}{dz^3} \frac{dz}{dx} \\ &= \frac{2}{x^3} \frac{dy}{dz} - \frac{1}{x^3} \frac{d^2 y}{dz^2} - \frac{2}{x^3} \frac{d^2 y}{dz^2} + \frac{1}{x^3} \frac{d^3 y}{dz^3} = \frac{1}{x^3} \frac{d^3 y}{dz^3} - \frac{3}{x^3} \frac{d^2 y}{dz^2} + \frac{2}{x^3} \frac{dy}{dz} \\ &= \frac{1}{x^3} [\mathcal{D}^3 - 3\mathcal{D}^2 + 2\mathcal{D}] y \longrightarrow x^3 \frac{d^3 y}{dx^3} = \mathcal{D} [\mathcal{D} - 1] [\mathcal{D} - 2] y \end{aligned} \quad (23)$$

From those, it could be said that Eq.(20) is the general solution.

Example 5.1. Solve the following differential equation, $x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4$. Since the characteristic of the function is the same as that of in Eq.(19), the question could be modified into $\mathcal{D} [\mathcal{D} - 1] y - 2\mathcal{D} y - 4y = e^{4z}$,

1) Complementary y_c : $z = \ln(x)$

$$[\mathcal{D}^2 - 3\mathcal{D} - 4] = 0 \longrightarrow \mathcal{D}_1 = 4, \mathcal{D}_2 = -1 \text{ such that } y_c = k_1 e^{4z} + k_2 e^{-z}$$

2) Particular y_p : Note that $e^{\pm iax} = \cos(ax) \pm i \sin(ax)$

$$\begin{aligned} y_p &= \left[\frac{1}{\mathcal{D}^2 - 3\mathcal{D} - 4} \right] e^{4z} = \left[\frac{1}{16 - 12 - 4} \right] e^{4z} = \frac{1}{0} e^{4z} \quad (\text{failed}) \\ &= \left[\frac{1}{\mathcal{D}^2 - 3\mathcal{D} - 4} \right] (e^{4z} \cdot 1) = e^{4z} \left[\frac{1}{(\mathcal{D} + 4)^2 - 3(\mathcal{D} + 4) - 4} \right] (1) = e^{4z} \left[\frac{1}{\mathcal{D}^2 + 5\mathcal{D}} \right] (1) \\ &= e^{4z} \left[\frac{1}{\mathcal{D} + 5} \right] \frac{1}{\mathcal{D}} (1) = e^{4z} \left[\frac{1}{5} - \frac{1}{25} \mathcal{D} \right] z = e^{4z} \left[\frac{1}{5} z - \frac{1}{25} \right] \\ &= x^4 \left(\frac{1}{5} \ln(x) - \frac{1}{25} \right) \rightarrow z = \ln(x) \end{aligned}$$

such that the final solution ($y = y_c + y_p$) is $k_1 x^4 + \frac{k_2}{x} + x^4 \left(\frac{1}{5} \ln x - \frac{1}{25} \right)$

Example 5.2. Find the solution of the following function, $x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = \ln(x)$! The question could be formed as Eq.(19), such that $\mathcal{D}[\mathcal{D} - 1]y + 2\mathcal{D}y = z$,

1) Complementary y_c : $z = \ln(x)$

$$[\mathcal{D}^2 + \mathcal{D}] = 0 \longrightarrow \mathcal{D}_1 = 0, \mathcal{D}_2 = -1 \text{ such that } y_c = k_1 + k_2 e^{-z}$$

2) Particular y_p : $z = \ln(\varphi)$ and $\varphi = (1 + x)$

$$\begin{aligned} y_p &= \left[\frac{1}{\mathcal{D}^2 + \mathcal{D}} \right] z = \left[\frac{1}{\mathcal{D} + 1} \right] \frac{1}{\mathcal{D}} z = \left[\frac{1}{\mathcal{D} + 1} \right] \frac{1}{2} z^2 \\ &= [1 - \mathcal{D} + \mathcal{D}^2] \frac{1}{2} z^2 = \frac{1}{2} z^2 - z + 1 = \frac{1}{2} \ln^2(x) - \ln(x) + 1 \end{aligned}$$

such that the final solution ($y = y_c + y_p$) is $k_1 + \frac{k_2}{x} + \frac{1}{2} \ln^2(x) - \ln(x) + 1$

Example 5.3. Supposed a function, $(1 + x)^2 \frac{d^2 y}{dx^2} + (1 + x) \frac{dy}{dx} + y = \sin(2 \ln(1 + x))$, find the solution! If $(1 + x) = \varphi \rightarrow dx = d\varphi$, then the equation is the same as that of Eq.(19), so that Eq.(20) follows with $\mathcal{D}[\mathcal{D} - 1]y + \mathcal{D}y + y = \sin(2z)$, therefore

1) Complementary y_c : $z = \ln(\varphi)$

$$[\mathcal{D}^2 + 1] = 0 \longrightarrow \mathcal{D}_1 = i, \mathcal{D}_2 = -i \text{ such that } y_c = k_1 e^{iz} + k_2 e^{-iz}$$

2) Particular y_p : $z = \ln(\varphi)$ and $\varphi = (1 + x)$

$$\begin{aligned} y_p &= \left[\frac{1}{\mathcal{D}^2 + 1} \right] \sin(2z) = \left[\frac{1}{-4 + 1} \right] \sin(2z) = -\frac{1}{3} \sin(2z) \\ &= -\frac{1}{3} \sin(2 \ln(1 + x)) \end{aligned}$$

such that the final solution ($y = y_c + y_p$) is $k \cos(\ln(1 + x) - \theta) - \frac{1}{3} \sin(2 \ln(1 + x))$

Example 5.4. Given a differential equation, $x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} - 3y = x^2$, obtain the solution! Since the characteristic is on par with Eq.(19), then the solutions are, $\mathcal{D}[\mathcal{D} - 1][\mathcal{D} - 2]y + 2\mathcal{D}[\mathcal{D} - 1]y + 3\mathcal{D}y - 3y = e^{2z}$

1) Complementary y_c : $z = \ln(x)$

$$\begin{aligned} [\mathcal{D}^3 - \mathcal{D}^2 + 3\mathcal{D} - 3] &= 0 \longrightarrow [\mathcal{D}^2 + 3][\mathcal{D} - 1] = 0 \\ \mathcal{D}_1 &= i\sqrt{3}, \mathcal{D}_2 = -i\sqrt{3}, \mathcal{D}_3 = 1 \text{ such that } y_c = k_1 e^{i\sqrt{3}z} + k_2 e^{-i\sqrt{3}z} + k_3 e^z \end{aligned}$$

2) Particular y_p : Note that $e^{\pm iax} = \cos(ax) \pm i \sin(ax)$

$$y_p = \left[\frac{1}{\mathcal{D}^3 - \mathcal{D}^2 + 3\mathcal{D} - 3} \right] e^{2z} = \left[\frac{1}{8 - 4 + 6 - 3} \right] e^{2z} = \frac{1}{7} e^{2z} = \frac{1}{7} x^2$$

such that the final solution ($y = y_c + y_p$) is $k \cos(\sqrt{3} \ln(x) - \theta) + k_3 e^z + \frac{1}{7} x^2$

Exercise 6. Solve the following differential equations!

1. $x^2 \frac{d^2 y}{dx^2} + y = 3x^2$	3. $x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = x^2 + \sin(5 \ln(x))$
2. $x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = \frac{42}{x^4}$	4. $x^2 \frac{d^3 y}{dx^3} + 3x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = x^2 \ln(x)$

6 Solution of Simultaneous Differential Equations

In this section, there exists some dependent variables φ_i from $1 \rightarrow n$, such as (x, y, z) or (p, q, r, s) if $n = 3$ or 4 in turn, paralleling with the number of equations, with respect to arbitrary independent variable (t) . The goal is then to find the solution of such dependent variables by eliminating one from another using Gauss elimination so that the ways to obtain the final solution ($\varphi_i = \varphi_c + \varphi_p$) depends on the number of equations.

Example 6.1. Find the solution of the given two equations with $x(0) = 2$ and $y(0) = 0$!

$$\frac{dx}{dt} + y = \sin(t) \quad \text{and} \quad x + \frac{dy}{dt} = \cos(t)$$

Since there are two equations, then there are two solutions with respect to dependent variables (x, y) with respect to independent (t) . In order to do that, Gauss elimination is applied in terms of certain dependent variable. The first is to multiply the second equation by \mathcal{D} and subtract it from Eq.(1), such that

$$(1) \quad \left. \begin{array}{l} \mathcal{D}x + y = \sin(t) \xrightarrow{\times 1} \mathcal{D}x + y = \sin(t) \\ x + \mathcal{D}y = \cos(t) \xrightarrow{\times \mathcal{D}} \mathcal{D}x + \mathcal{D}^2 y = \mathcal{D}[\cos(t)] \end{array} \right\} (-) [1 - \mathcal{D}^2] y = 2 \sin(t)$$

1) Complementary y_c :

$$[1 - \mathcal{D}^2] = 0 \longrightarrow \mathcal{D}_1 = 1, \mathcal{D}_2 = -1 \text{ such that } y_c = k_1 e^t + k_2 e^{-t}$$

2) Particular y_p :

$$y_p = \left[\frac{1}{1 - \mathcal{D}^2} \right] 2 \sin(t) = \frac{1}{1 - (-1)} 2 \sin(t) = \sin(t)$$

such that the solution with respect to y is $k_1 e^t + k_2 e^{-t} + \sin(t)$. If the constraint is applied, then it results in

$$y(0) : k_1 + k_2 = 0 \longrightarrow k_2 = -k_1$$

The second formulation is then to remove y so that it yields x , with the same formula, it has

$$(2) \quad \left. \begin{array}{l} \mathcal{D}x + y = \sin(t) \xrightarrow{\times \mathcal{D}} \mathcal{D}^2 x + \mathcal{D}y = \mathcal{D}[\sin(t)] \\ x + \mathcal{D}y = \cos(t) \xrightarrow{\times 1} x + \mathcal{D}y = \cos(t) \end{array} \right\} (-) [\mathcal{D}^2 - 1] x = 0$$

1) Complementary x_c :

$$[1 - \mathcal{D}^2] = 0 \longrightarrow \mathcal{D}_1 = 1, \mathcal{D}_2 = -1 \text{ such that } x_c = c_1 e^t + c_2 e^{-t}$$

2) Particular x_p : Since the right-hand side is zero, there is no x_p and the solution depends solely on $x = x_c$. Applying the constraint, it constitutes

$$x(0) : c_1 + c_2 = 2 \longrightarrow c_2 = 2 - c_1$$

To gain the constant parameters from the constraints, those two results should be compared into the given questions, yet we should have the first derivative of the two, therefore

$$\frac{dy}{dt} = k_1 e^t + k_2 e^{-t} + \cos(t) \quad \text{and} \quad \frac{dx}{dt} = c_1 e^t + (c_1 - 2) e^{-t}$$

such that, taking the first equation $\mathcal{D}x + y = \sin(t)$, it gets

$$c_1 e^t + (c_1 - 2) e^{-t} + k_1 e^t - k_2 e^{-t} + \sin(t) = \sin(t) \longrightarrow c_1 + k_1 = 0 \text{ and } c_1 - 2 - k_2 = 0$$

From the final equation it then can be concluded that $k_1 = -1, k_2 = 1, c_1 = 1$ and $c_2 = 1$. Keep in mind that using Eq.(2) in the question will lead to the same answer, which means that

$$y = e^{-t} - e^t + \sin(t) \quad \text{and} \quad x = e^t + e^{-t}$$

Note: If there are no certain constraints mentioned in the question, then the final solutions halt in the certain dependent variables without doing further comparison to fulfill the constraints.

Example 6.2. Solve the following two simultaneous differential equations without constraints!

$$2 \frac{dx}{dt} + 6x - y = 2 \sin(2t) \quad \text{and} \quad \frac{dy}{dt} - 2x + 5y = 0$$

The first is to multiply the second equation by $2\mathcal{D} + 6$ and subtract it from Eq.(1), such that

$$(1) \quad \begin{array}{rcl} [2\mathcal{D} + 6] x - y = 2 \sin(2t) & \xrightarrow{\times 2} & 2 [2\mathcal{D} + 6] x - 2y = 4 \sin(2t) \\ -2x + [\mathcal{D} + 5] y = 0 & \xrightarrow{\times (2\mathcal{D} + 6)} & -2 [2\mathcal{D} + 6] x + [2\mathcal{D}^2 + 16\mathcal{D} + 30] y = 0 \end{array}$$

1) Complementary y_c : Adding the two equations and using (abc) -formula result in

$$[2\mathcal{D}^2 + 16\mathcal{D} + 28] = 0 \longrightarrow \mathcal{D}_{1,2} = -4 \pm 2\sqrt{2} \text{ such that } y_c = k_1 e^{(-4+2\sqrt{2})t} + k_2 e^{(-4-2\sqrt{2})t}$$

2) Particular y_p :

$$\begin{aligned} y_p &= \left[\frac{1}{\mathcal{D}^2 + 8\mathcal{D} + 14} \right] 2 \sin(2t) = \left[\frac{1}{-4 + 8\mathcal{D} + 14} \right] 2 \sin(2t) = \left[\frac{1}{4\mathcal{D} + 5} \right] \sin(2t) \\ &= \left[\frac{4\mathcal{D} - 5}{16\mathcal{D}^2 - 25} \right] \sin(2t) = -\frac{1}{89} [4\mathcal{D} - 5] \sin(2t) = -\frac{1}{89} [8 \cos(2t) - 5 \sin(2t)] \end{aligned}$$

such that the solution with respect to y is $k_1 e^{(-4+2\sqrt{2})t} + k_2 e^{(-4-2\sqrt{2})t} - \frac{1}{89} [8 \cos(2t) - 5 \sin(2t)]$. The second formulation is then to remove y so that it yields x , with the same formula, it has

$$(2) \quad \begin{array}{rcl} [2\mathcal{D} + 6] x - y = 2 \sin(2t) & \xrightarrow{\times (\mathcal{D} + 5)} & [2\mathcal{D} + 6] [\mathcal{D} + 5] x - [\mathcal{D} + 5] y = [\mathcal{D} + 5] 2 \sin(2t) \\ -2x + [\mathcal{D} + 5] y = 0 & \xrightarrow{\times 1} & -2x + [\mathcal{D} + 5] y = 0 \end{array}$$

1) Complementary x_c : Adding the two equations and using (abc) -formula result in

$$[2\mathcal{D}^2 + 16\mathcal{D} + 28] = 0 \longrightarrow \mathcal{D}_{1,2} = -4 \pm 2\sqrt{2} \text{ such that } x_c = c_1 e^{(-4+2\sqrt{2})t} + c_2 e^{(-4-2\sqrt{2})t}$$

2) Particular x_p :

$$\begin{aligned} x_p &= \left[\frac{1}{\mathcal{D}^2 + 8\mathcal{D} + 14} \right] [4 \cos(2t) + 10 \sin(2t)] = \left[\frac{1}{4\mathcal{D} + 5} \right] [2 \cos(2t) + 5 \sin(2t)] \\ &= \left[\frac{4\mathcal{D} - 5}{16\mathcal{D}^2 - 25} \right] [2 \cos(2t) + 5 \sin(2t)] = -\frac{1}{89} [4\mathcal{D} - 5] [2 \cos(2t) + 5 \sin(2t)] \\ &= -\frac{1}{89} [-16 \sin(2t) - 10 \cos(2t) + 40 \cos(2t) - 25 \sin(2t)] = \frac{1}{89} [41 \sin(2t) - 30 \cos(2t)] \end{aligned}$$

such that, the final solution with respect to x is in the following.

$$x = x_c + x_p \longrightarrow c_1 e^{(-4+2\sqrt{2})t} + c_2 e^{(-4-2\sqrt{2})t} + \frac{1}{89} [41 \sin(2t) - 30 \cos(2t)]$$

Exercise 7. Solve the following differential equations!

- | | |
|--|--|
| <p>1. $\frac{dx}{dt} + 5x - 2y = t$ and $\frac{dy}{dt} + 2x + y = 0$
with $x(0) = 0, y(0) = 0$</p> <p>2. $t \frac{dx}{dt} + y = t$ and $t \frac{dy}{dt} + x = 0$
with $x(1) = 1, y(-1) = 0$</p> <p>3. $\frac{dx}{dt} = y - z, \frac{dy}{dt} = z - x, \frac{dz}{dt} = x - y$</p> | <p>with $x(0) = 0, y(0) = 3, z(0) = 0$</p> <p>4. Given two vertical series of spring $K_1 = 5$ N/m and $K_2 = 10$ N/m with mass $M = 1$ kg in the top. x_1, x_2, y are the deviation in M, in between of springs and bottom in turn. Find x_1, x_2!</p> |
|--|--|

7 Solution of Partial Differential Equations (Linear Homogeneous)

Partial Differential Equation (PDE) has a function (φ) with more than one independent variables, typically two (x, y) in this section. Due to the more independent variables, it is required to differ the writing in terms of \mathcal{D} -term with respect to certain independent variable, such that

$$\frac{\partial \varphi(x, y)}{\partial x} = \mathcal{D}_x \varphi(x, y) \quad \frac{\partial^2 \varphi(x, y)}{\partial y^2} = \mathcal{D}_y^2 \varphi(x, y) \quad \frac{\partial^2 \varphi(x, y)}{\partial x \partial y} = \mathcal{D}_x \mathcal{D}_y \varphi(x, y) \quad (24)$$

This section focuses on the linear, as a_i with $i = 0 \rightarrow n$ is in the first-order, and homogeneous, due to the order of each term is the same n , function as written in the following,

$$a_0 \frac{\partial^n \varphi}{\partial x^n} + a_1 \frac{\partial^n \varphi}{\partial x^{n-1} \partial y} + \cdots + a_n \frac{\partial^n \varphi}{\partial y^n} = \Phi(x, y) \quad (25)$$

$$[a_0 \mathcal{D}_x^n + a_1 \mathcal{D}_x^{n-1} \mathcal{D}_y + \cdots + a_n \mathcal{D}_y^n] \varphi = \Phi(x, y) \longrightarrow \Psi(\mathcal{D}_x, \mathcal{D}_y) \varphi = \Phi(x, y) \quad (26)$$

Supposed the partial differential function Eq.(25) with order $n = 2$, the steps to gain the solution are presented in the following. To achieve complementary solution φ_c , the right-hand side function $\Phi(x, y)$ is assumed to be zero and the function is made of in terms of \mathcal{D} ,

$$a_0 \frac{\partial^2 \varphi}{\partial x^2} + a_1 \frac{\partial^2 \varphi}{\partial x \partial y} + a_2 \frac{\partial^2 \varphi}{\partial y^2} = 0 \longrightarrow \underbrace{[a_0 \mathcal{D}_x^2 + a_1 \mathcal{D}_x \mathcal{D}_y + a_2 \mathcal{D}_y^2]}_{\Psi(\mathcal{D}_x, \mathcal{D}_y)} = 0 \quad (27)$$

Next, take $\mathcal{D}_x = m$ and $\mathcal{D}_y = 1$ and find the roots of the modified function,

$$a_0 m^2 + a_1 m + a_2 = 0 \longrightarrow m_{(1,2)} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}$$

The last is to form the complementary solution depending on m_1 and m_2 . Keep in mind that the solution is not unique as long as it fulfills the equation in φ_c

$$\varphi_c = \Phi_1(y + m_1 x) + \Phi_2(y + m_2 x) \longrightarrow \text{if } m_1 \neq m_2 \quad (28)$$

$$\varphi_c = \Phi_1(y + mx) + x \Phi_2(y + mx) \longrightarrow \text{if } m_1 = m_2 \quad (29)$$

To achieve particular solution φ_p , there are some possible ways depending on function $\Phi(x, y)$

1) If $\Phi(x, y) = e^{px+qy}$ and $\Psi(\mathcal{D}_x, \mathcal{D}_y)$ with $n = 2 \longrightarrow a_0 \mathcal{D}_x^2 + a_1 \mathcal{D}_x \mathcal{D}_y + a_2 \mathcal{D}_y^2$

$$\varphi_p = \left[\frac{1}{a_0 \mathcal{D}_x^2 + a_1 \mathcal{D}_x \mathcal{D}_y + a_2 \mathcal{D}_y^2} \right] e^{px+qy} = \left[\frac{1}{\Psi(\mathcal{D}_x, \mathcal{D}_y)} \right] e^{px+qy} \longrightarrow \left[\frac{1}{\Psi(p, q)} \right] e^{px+qy}$$

Proof.

$$\begin{aligned} \Psi(\mathcal{D}_x, \mathcal{D}_y) \Phi(x, y) &= a_0 [\mathcal{D}_x^2] e^{px+qy} + a_1 [\mathcal{D}_x \mathcal{D}_y] e^{px+qy} + a_2 [\mathcal{D}_y^2] e^{px+qy} \\ &= a_0 (p^2) e^{px+qy} + a_1 (pq) e^{px+qy} + a_2 (q^2) e^{px+qy} \\ &= (a_0 p^2 + a_1 pq + a_2 q^2) e^{px+qy} \longrightarrow \Psi(p, q) e^{px+qy} \quad \blacksquare \end{aligned}$$

2) If $\Phi(x, y) = \sin(px + qy)$ or $\cos(px + qy)$ and $\Psi(\mathcal{D}_x, \mathcal{D}_y)$ with $n = 2$

$$\varphi_p = \left[\frac{1}{\Psi(\mathcal{D}_x^2, \mathcal{D}_x \mathcal{D}_y, \mathcal{D}_y^2)} \right] \sin(px + qy) \longrightarrow \left[\frac{1}{\Psi(-p^2, -pq, -q^2)} \right] \sin(px + qy)$$

Proof.

$$\begin{aligned} \Psi(\mathcal{D}_x, \mathcal{D}_y) \Phi(x, y) &= a_0 [\mathcal{D}_x^2] \sin(px + qy) + a_1 [\mathcal{D}_x \mathcal{D}_y] \sin(px + qy) + a_2 [\mathcal{D}_y^2] \sin(px + qy) \\ &= a_0 (-p^2) \sin(px + qy) + a_1 (-pq) \sin(px + qy) + a_2 (-q^2) \sin(px + qy) \\ &= \underbrace{[a_0 (-p^2) + a_1 (-pq) + a_2 (-q^2)]}_{\Psi(-p^2, -pq, -q^2) \neq 0} \sin(px + qy) \quad \blacksquare \end{aligned}$$

3) If $\Phi(x, y) = x^p y^q$ or $(x + y)^r$ and $\Psi(\mathcal{D}_x, \mathcal{D}_y)$ with $n = 2$. It supposed to be in series

$$\begin{aligned}\varphi_p &= \left[\frac{1}{a_0 \mathcal{D}_x^2 + a_1 \mathcal{D}_x \mathcal{D}_y + a_2 \mathcal{D}_y^2} \right] x^p y^q = \frac{1}{\mathcal{D}_x^2} \left[\frac{1}{a_0 + a_1 \left(\frac{\mathcal{D}_y}{\mathcal{D}_x} \right) + a_2 \left(\frac{\mathcal{D}_y}{\mathcal{D}_x} \right)^2} \right] x^p y^q \rightarrow \frac{\mathcal{D}_y}{\mathcal{D}_x} = u \\ &= \frac{1}{\mathcal{D}_x^2} \left[\frac{1}{a_0} + \left(\frac{-a_1}{a_0^2} \right) u + \left(\frac{-a_2}{a_0^2} + \frac{a_1^2}{a_0^3} \right) u^2 + \dots \text{stop when } n = q \right] x^p y^q\end{aligned}$$

Example 7.1. Find the solution of this partial function, $\frac{\partial^2 \varphi}{\partial x^2} + 4 \frac{\partial^2 \varphi}{\partial x \partial y} - 5 \frac{\partial^2 \varphi}{\partial y^2} = 0$. The given function is altered into $[\mathcal{D}_x^2 + 4\mathcal{D}_x \mathcal{D}_y - 5\mathcal{D}_y^2] \varphi = 0$ and the solution is

1) Complementary φ_c : $\mathcal{D}_x = m$ and $\mathcal{D}_y = 1$

$$m^2 + 4m - 5 = 0 \longrightarrow m_1 = 1, m_2 = -5 \text{ such that } \varphi_c = \Phi_1(y + x) + \Phi_2(y - 5x)$$

2) Particular φ_p : Since $\varphi_p = 0$, the solution is $\varphi = \varphi_c$ and one of the solutions fulfilling comprises

$$\varphi = (y + x)^2 + (y - 5x)^3 \quad (30)$$

Bear in mind that the solution is not unique, and one of the instances fulfilling the equation in the solution is mentioned in Eq.(30). To prove that the solution is correct, we need to bring back into the question, such that

$$\frac{\partial \varphi}{\partial x} = 2(y + x) - 15(y - 5x)^2 \longrightarrow \frac{\partial^2 \varphi}{\partial x^2} = 2 + 150y - 750x \quad (31)$$

$$\frac{\partial \varphi}{\partial y} = 2(y + x) + 3(y - 5x)^2 \longrightarrow \frac{\partial^2 \varphi}{\partial x^2} = 2 + 6y - 30x \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial x \partial y} = 2 - 30y + 150x \quad (32)$$

If Eqs.(31) and (32) are brought to the question, it results in

$$\begin{aligned}2 + 150y - 750x + 4(2 - 30y + 150x) - 5(2 + 6y - 30x) &= 0 \\ 2 + 8 - 10 + 150y - 120y - 30y - 750x + 600x + 150x &= 0\end{aligned}$$

Example 7.2. Solve this partial differential equation, $2 \frac{\partial^2 \varphi}{\partial x^2} + 5 \frac{\partial^2 \varphi}{\partial x \partial y} + 2 \frac{\partial^2 \varphi}{\partial y^2} = 0$. The question is transformed into $[2\mathcal{D}_x^2 + 5\mathcal{D}_x \mathcal{D}_y + 2\mathcal{D}_y^2] \varphi = 0$ and the solution is

1) Complementary φ_c : $\mathcal{D}_x = m$ and $\mathcal{D}_y = 1$

$$2m^2 + 5m + 2 = 0 \longrightarrow m_1 = -\frac{1}{2}, m_2 = -2 \text{ such that } \varphi_c = \Phi_1\left(y - \frac{1}{2}x\right) + \Phi_2(y - 2x)$$

2) Particular φ_p : Since $\varphi_p = 0$, the solution is $\varphi = \varphi_c$ and one of the solutions fulfilling comprises

$$\varphi = \sqrt{2y + x} + \sin(y - 2x)$$

Example 7.3. Solve this partial differential equation, $\frac{\partial^2 \varphi}{\partial x^2} - 5 \frac{\partial^2 \varphi}{\partial x \partial y} + 6 \frac{\partial^2 \varphi}{\partial y^2} = e^{x+4y}$. In terms of \mathcal{D} -term, the question could be written as $[\mathcal{D}_x^2 - 5\mathcal{D}_x \mathcal{D}_y + 6\mathcal{D}_y^2] \varphi = e^{x+4y}$

1) Complementary φ_c : $\mathcal{D}_x = m$ and $\mathcal{D}_y = 1$

$$m^2 - 5m + 6 = 0 \longrightarrow m_1 = 2, m_2 = 3 \text{ such that } \varphi_c = \Phi_1(y + 2x) + \Phi_2(y + 3x)$$

2) Particular φ_p : $\mathcal{D}_x = 1$ and $\mathcal{D}_y = 4$

$$\varphi_p = \left[\frac{1}{\mathcal{D}_x^2 - 5\mathcal{D}_x \mathcal{D}_y + 6\mathcal{D}_y^2} \right] e^{x+4y} = \left[\frac{1}{1 - 20 + 96} \right] e^{x+4y} = \frac{1}{77} e^{x+4y}$$

such that, the final solution is $\varphi = \varphi_c + \varphi_p$ and one of the solutions fulfilling comprises

$$\varphi = \sqrt{y + 2x} + \cos(y + 3x) + \frac{1}{77} e^{x+4y}$$

Example 7.4. Given the function, $\frac{\partial^3 \varphi}{\partial x^3} - 4 \frac{\partial^3 \varphi}{\partial x^2 \partial y} + 4 \frac{\partial^3 \varphi}{\partial x \partial y^2} = 2 \sin(3x + 2y)$ and it could be simplified into $[\mathcal{D}_x^3 - 4\mathcal{D}_x^2 \mathcal{D}_y + 4\mathcal{D}_x \mathcal{D}_y^2] \varphi = 2 \sin(3x + 2y)$

1) Complementary φ_c : $\mathcal{D}_x = m$ and $\mathcal{D}_y = 1$

$$m^3 - 4m^2 + 4m = 0 \longrightarrow m_1 = 0, m_{2,3} = 2 \text{ such that } \varphi_c = \Phi_1(y) + \Phi_2(y + 2x) + x\Phi_2(y + 2x)$$

2) Particular φ_p : $\mathcal{D}_x^2 = -9$, $\mathcal{D}_x \mathcal{D}_y = -6$ and $\mathcal{D}_y^2 = -4$

$$\begin{aligned} \varphi_p &= \left[\frac{1}{\mathcal{D}_x^3 - 4\mathcal{D}_x^2 \mathcal{D}_y + 4\mathcal{D}_x \mathcal{D}_y^2} \right] 2 \sin(3x + 2y) = \frac{1}{\mathcal{D}_x} \left[\frac{2}{\mathcal{D}_x^2 - 4\mathcal{D}_x \mathcal{D}_y + 4\mathcal{D}_y^2} \right] \sin(3x + 2y) \\ &= \frac{1}{\mathcal{D}_x} \left[\frac{2}{-9 + 24 - 16} \right] \sin(3x + 2y) = -2 \left[\frac{1}{\mathcal{D}_x} \right] \sin(3x + 2y) = \frac{2}{3} \cos(3x + 2y) \end{aligned}$$

such that, the final solution is $\varphi = \varphi_c + \varphi_p$ and one of the solutions fulfilling comprises

$$y + (y + 2x)^2 + x(y + 2x) + \frac{2}{3} \cos(3x + 2y)$$

Example 7.5. Supposed a partial function, $\frac{\partial^2 \varphi}{\partial x^2} - 8 \frac{\partial^2 \varphi}{\partial x \partial y} + 15 \frac{\partial^2 \varphi}{\partial y^2} = xy^2$, find the solution!

First, it is necessary to change the term, $[\mathcal{D}_x^2 - 8\mathcal{D}_x \mathcal{D}_y + 15\mathcal{D}_y^2] \varphi = xy^2$ and the solutions are,

1) Complementary φ_c : $\mathcal{D}_x = m$ and $\mathcal{D}_y = 1$

$$m^2 - 8m + 15 = 0 \longrightarrow m_1 = 5, m_2 = 3 \text{ such that } \varphi_c = \Phi_1(y + 5x) + \Phi_2(y + 3x)$$

2) Particular φ_p :

$$\begin{aligned} \varphi_p &= \frac{1}{\mathcal{D}_x^2} \left[\frac{1}{1 - 8 \left(\frac{\mathcal{D}_y}{\mathcal{D}_x} \right) + 15 \left(\frac{\mathcal{D}_y}{\mathcal{D}_x} \right)^2} \right] xy^2 = \left[\frac{1}{1 - 8u + 15u^2} \right] xy^2 \longrightarrow \frac{\mathcal{D}_y}{\mathcal{D}_x} = u \\ &= \frac{1}{\mathcal{D}_x^2} \left[1 - 8 \left(\frac{\mathcal{D}_y}{\mathcal{D}_x} \right) + 49 \left(\frac{\mathcal{D}_y}{\mathcal{D}_x} \right)^2 + \dots \right] xy^2 \quad (\text{stop when } n = q) \\ &= \left[\frac{1}{\mathcal{D}_x^2} \right] \left(xy^2 - 8x^2y + \frac{49}{3}x^3 \right) = \frac{1}{6}x^3y^2 - \frac{2}{3}x^4y + \frac{49}{60}x^5 \end{aligned}$$

such that, the final solution is $\varphi = \varphi_c + \varphi_p$ and one of the solutions fulfilling comprises

$$(y + 5x)^3 + \tan(y + 3x) + \frac{1}{6}x^3y^2 - \frac{2}{3}x^4y + \frac{49}{60}x^5$$

Exercise 8. Solve the following differential equations!

$$\left. \begin{array}{l} 1. \frac{\partial^2 \varphi}{\partial x^2} - 2 \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \\ 2. \frac{\partial^3 \varphi}{\partial x^3} - 6 \frac{\partial^3 \varphi}{\partial x^2 \partial y} + 11 \frac{\partial^3 \varphi}{\partial x \partial y^2} - 6 \frac{\partial^3 \varphi}{\partial y^3} = 0 \\ 3. \frac{\partial^3 \varphi}{\partial x^3} - 6 \frac{\partial^3 \varphi}{\partial x^2 \partial y} + 12 \frac{\partial^3 \varphi}{\partial x \partial y^2} - 8 \frac{\partial^3 \varphi}{\partial y^3} = 0 \end{array} \right\} \begin{array}{l} 4. \frac{\partial^4 \varphi}{\partial x^4} + \frac{\partial^4 \varphi}{\partial y^4} = 0 \\ 5. \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial x \partial y} = 2 \sin(x) \cos(2y) \\ 6. \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial x \partial y} - 6 \frac{\partial^2 \varphi}{\partial y^2} = x + y \end{array}$$

It is possible that the three possible ways in finding the particular solution are failed because of zero result in denominator. If this really happens, it is required to conduct what is known as Parameter Variation method, such that

$$\left[\frac{1}{\mathcal{D}_x - a\mathcal{D}_y} \right] \Phi(x, y) = \int \Phi(x, c - ax) dx \longrightarrow y = c - ax \quad (33)$$

where a and c are the constant parameter of \mathcal{D}_y and an integration constant respectively. Keep in mind that,

$$\frac{\partial z}{\partial x} - a \frac{\partial z}{\partial y} = \Phi(x, y) \longrightarrow z = \left[\frac{1}{\mathcal{D}_x - a\mathcal{D}_y} \right] \Phi(x, y) \quad (34)$$

is a Lagrange differential equation, and to solve, it needs a auxiliary equation, therefore

$$\frac{dx}{1} = \frac{dy}{-a} = \frac{dz}{\Phi(x, y)} \quad (35)$$

$$\int -a dx = \int dy \rightarrow y = -ax + c \quad \text{and} \quad \int dz = \int \Phi(x, y) dx \rightarrow z = \int \Phi(x, c - ax) dx \quad (36)$$

Example 7.6. Solve the following partial differential equation, $\frac{\partial^2 \varphi}{\partial x^2} - 2 \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial y^2} = e^{x+y}$

1) Complementary φ_c : $\mathcal{D}_x = m$ and $\mathcal{D}_y = 1$

$$m^2 - 2m + 1 = 0 \longrightarrow m_1 = 1, m_2 = 1 \text{ such that } \varphi_c = \Phi_1(y + x) + x\Phi_2(y + x)$$

2) Particular φ_p : $\mathcal{D}_x = 1$ and $\mathcal{D}_y = 1$

$$\begin{aligned} \varphi_p &= \left[\frac{1}{\mathcal{D}_x^2 - 2\mathcal{D}_x\mathcal{D}_y + \mathcal{D}_y^2} \right] e^{x+y} = \frac{1}{0} e^{x+y} \quad (\text{failed}) \\ &= \left[\frac{1}{\mathcal{D}_x - \mathcal{D}_y} \right] \underbrace{\left[\frac{1}{\mathcal{D}_x - \mathcal{D}_y} \right] e^{x+y}}_A \longrightarrow A = \int e^c dx = x e^c \quad (y = c - x \rightarrow e^{x+y} = e^c) \\ &= \left[\frac{1}{\mathcal{D}_x - \mathcal{D}_y} \right] x e^{x+y} = \int x e^c dx = \frac{1}{2} x^2 e^c \quad (y = c - x \rightarrow x e^{x+y} = x e^c) \end{aligned}$$

such that, one of the final solutions is $\varphi = (y + x)^2 + x \tan(y + x) + \frac{1}{2} x^2 e^{x+y}$

Example 7.7. Given the \mathcal{D} -term partial function, $[\mathcal{D}_x - \mathcal{D}_y][\mathcal{D}_x + 2\mathcal{D}_y]\varphi = (y + 1)e^x$

1) Complementary φ_c : $\mathcal{D}_x = m$ and $\mathcal{D}_y = 1$

$$(m - 1)(m + 2) = 0 \longrightarrow m_1 = 1, m_2 = -2 \text{ such that } \varphi_c = \Phi_1(y + x) + \Phi_2(y - 2x)$$

2) Particular φ_p : $\mathcal{D}_x = 1$ and $\mathcal{D}_y = 1$

$$\begin{aligned} \varphi_p &= \underbrace{\left[\frac{1}{\mathcal{D}_x - \mathcal{D}_y} \right]}_B \underbrace{\left[\frac{1}{\mathcal{D}_x + 2\mathcal{D}_y} \right] (y + 1)e^x}_{A} = y e^x, \text{ where} \\ A &= \int (c + 2x + 1)e^x dx = c e^x + 2x e^x - e^x = y e^x - e^x \quad (y = c + 2x \rightarrow (c + 2x + 1)e^x) \\ B &= \int (c - x)e^x - e^x dx = c e^x - x e^x = y e^x \quad (y = c - x \rightarrow (c - x)e^x - e^x) \end{aligned}$$

such that, the final solution is $\varphi = (y + x) + (y - 2x)^2 + y e^x$

Exercise 9. Solve the following differential equations!

$$1. \frac{\partial^2 \varphi}{\partial x^2} + 2 \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial y^2} = 2 \sin(y) - x \cos(y) \quad \left| \quad 2. \frac{\partial^3 \varphi}{\partial x^3} - 2 \frac{\partial^3 \varphi}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y \right.$$

8 Solution of Partial Differential Equations (Linear Non-Homogeneous)

This chapter delivers the partial function yet homogeneous. Compared to the preceding, it gives different order of the derivative terms as shown in the following instance, such that

$$a_0 \frac{\partial^n \varphi}{\partial x^n} + a_1 \frac{\partial^n \varphi}{\partial x \partial y} + a_2 \frac{\partial^k \varphi}{\partial y^k} + a_3 \frac{\partial \varphi}{\partial x} + a_4 \frac{\partial^m \varphi}{\partial y^m} + a_5 \varphi = 0 \longrightarrow \Psi(\mathcal{D}_x, \mathcal{D}_y) = 0 \quad (37)$$

where a_0, a_1, \dots, a_5 are in the same order, then called as linear, and n, k, m are different then it is supposed to be non-homogeneous. The solutions constitutes complementary and particular. To solve complementary solution (φ_c), it is necessary to form as follows,

$$[\mathcal{D}_x - m\mathcal{D}_y - a] z = 0 \longrightarrow \frac{\partial z}{\partial x} - m \frac{\partial z}{\partial y} - az = 0 \quad (38)$$

Since it is a Lagrange differential equation, and to solve, it needs a auxiliary equation, therefore

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{az} \quad (39)$$

$$\int -a dx = \int dy \rightarrow y = -mx + c \quad \text{and} \quad \int \frac{1}{az} dz \rightarrow ax = \ln(z) \quad (40)$$

such that,

$$\varphi_c = \Phi_1(y + m_1 x) e^{ax} + \Phi_2(y + m_2 x) e^{ax} \longrightarrow \text{if } m_1 \neq m_2 \quad (41)$$

$$\varphi_c = [\Phi_1(y + mx) + x\Phi_2(y + mx)] e^{ax} \longrightarrow \text{if } m_1 = m_2 \quad (42)$$

To achieve particular solution φ_p , there are some possible ways depending on function $\Phi(x, y)$. With the same definition of $\Psi(\mathcal{D}_x, \mathcal{D}_y)$, it can be enlightened that

1) If $\Phi(x, y) = e^{px+qy}$

$$\varphi_p = \left[\frac{1}{\Psi(\mathcal{D}_x, \mathcal{D}_y)} \right] e^{px+qy} \longrightarrow \left[\frac{1}{\Psi(p, q)} \right] e^{px+qy}$$

2) If $\Phi(x, y) = \sin(px + qy)$ or $\cos(px + qy)$

$$\varphi_p = \left[\frac{1}{\Psi(\mathcal{D}_x^2, \mathcal{D}_x \mathcal{D}_y, \mathcal{D}_y^2)} \right] \sin(px + qy) \longrightarrow \left[\frac{1}{\Psi(-p^2, -pq, -q^2)} \right] \sin(px + qy)$$

3) If $\Phi(x, y) = x^p y^q$ or $(x + y)^r$, it supposed that $\Psi(\mathcal{D}_x, \mathcal{D}_y)$ as used in the example and the form would be more various and it should be arranged in series

$$\begin{aligned} \varphi_p &= \left[\frac{1}{a_0 \mathcal{D}_x^2 + a_1 \mathcal{D}_x \mathcal{D}_y + a_2 \mathcal{D}_y^2} \right] x^p y^q = \frac{1}{\mathcal{D}_x^2} \left[\frac{1}{a_0 + a_1 \left(\frac{\mathcal{D}_y}{\mathcal{D}_x} \right) + a_2 \left(\frac{\mathcal{D}_y}{\mathcal{D}_x} \right)^2} \right] x^p y^q \rightarrow \frac{\mathcal{D}_y}{\mathcal{D}_x} = u \\ &= \frac{1}{\mathcal{D}_x^2} \left[\frac{1}{a_0} + \left(\frac{-a_1}{a_0^2} \right) u + \left(\frac{-a_2}{a_0^2} + \frac{a_1^2}{a_0^3} \right) u^2 + \dots \text{stop when } n = q \right] x^p y^q \end{aligned}$$

4) If $\Phi(x, y) = e^{px+qy} \phi(x, y)$

$$\varphi_p = \left[\frac{1}{\Psi(\mathcal{D}_x, \mathcal{D}_y)} \right] e^{px+qy} \phi(x, y) \longrightarrow e^{px+qy} \left[\frac{1}{\Psi(\mathcal{D}_x + p, \mathcal{D}_y + q)} \right] \phi(x, y)$$

Those four ways denoted above are the most common solutions depending on the right-hand side function and non-zero.

Example 8.1. Solve this partial function, $[\mathcal{D}_x^2 + 5\mathcal{D}_x\mathcal{D}_y - 5\mathcal{D}_x + 4\mathcal{D}_y^2 - 11\mathcal{D}_y + 6]\varphi = 0$. The first is to split that function into Eq.(38), so that with the help of *abc* formula, the given question could be arranged as follows

$$\mathcal{D}_x^2 + \mathcal{D}_x \underbrace{(5\mathcal{D}_y - 5)}_b + \underbrace{4\mathcal{D}_y^2 - 11\mathcal{D}_y + 6}_c = 0 \quad \text{w.r.t } (\mathcal{D}_x)$$

such that,

$$\mathcal{D}_x = \frac{-(5\mathcal{D}_y - 5) \pm \sqrt{(5\mathcal{D}_y - 5)^2 - 4\mathcal{D}_y^2 - 11\mathcal{D}_y + 6}}{2}$$

$$\mathcal{D}_x = \frac{-(5\mathcal{D}_y - 5) \pm (3\mathcal{D}_y - 1)}{2} \longrightarrow [\mathcal{D}_x + \mathcal{D}_y - 2][\mathcal{D}_x + 4\mathcal{D}_y - 3]\varphi = 0$$

Compared to $[\mathcal{D}_x - m\mathcal{D}_y - a]$, the question constitutes $m_1 = -1, a_1 = 2$ and $m_2 = -4, a_1 = 3$ such that from the information of Eqs.(38)-(40) the complementary solution which is also the final solution due to $\varphi_p = 0$ is

$$\varphi_c = \Phi_1(y - x)e^{2x} + \Phi_2(y - 4x)e^{3x}$$

Example 8.2. Given the following partial equation, $[\mathcal{D}_x + 3\mathcal{D}_y + 4]^2\varphi = 0$, find the solution! Since the question is squared, there might be using Eq. (34), such that $m = -3, a = -4$ and the solution is

$$\varphi_c = [\Phi_1(y - 3x) + x\Phi_2(y - 3x)]e^{-4x}$$

Example 8.3. Solve this partial differential equation, $[\mathcal{D}_x - \mathcal{D}_y - 2][\mathcal{D}_x - \mathcal{D}_y - 3]\varphi = e^{3x-2y}$
1) Complementary φ_c :

$$[\mathcal{D}_x - \mathcal{D}_y - 2][\mathcal{D}_x - \mathcal{D}_y - 3] = 0 \longrightarrow m_1 = 1, a_1 = 2 \text{ and } m_2 = 1, a_2 = 3, \text{ such that}$$

$$\varphi_c = e^{2x}\Phi_1(y + x) + e^{3x}\Phi_2(y + x)$$

2) Particular φ_p :

$$\varphi_p = \left[\frac{1}{(\mathcal{D}_x - \mathcal{D}_y - 2)(\mathcal{D}_x - \mathcal{D}_y - 3)} \right] e^{3x-2y} = \left[\frac{1}{(3+2-2)(3+2-3)} \right] e^{3x-2y} = \frac{1}{6} e^{3x-2y}$$

such that the final solution $\varphi = \varphi_c + \varphi_p$ is

$$\varphi = e^{2x}\Phi_1(y + x) + e^{3x}\Phi_2(y + x) + \frac{1}{6}e^{3x-2y}$$

Example 8.4. Find the solution of this partial function, $[\mathcal{D}_x + 1][\mathcal{D}_x + \mathcal{D}_y - 1]\varphi = \sin(x + 2y)$
1) Complementary φ_c :

$$[\mathcal{D}_x + 1][\mathcal{D}_x + \mathcal{D}_y - 1] = 0 \longrightarrow m_1 = 0, a_1 = -1 \text{ and } m_2 = -1, a_2 = 1, \text{ such that}$$

$$\varphi_c = e^{-x}\Phi_1(y) + e^x\Phi_2(y - x)$$

2) Particular φ_p : $[\mathcal{D}_x + 1][\mathcal{D}_x + \mathcal{D}_y - 1] = \mathcal{D}_x^2 + \mathcal{D}_x\mathcal{D}_y + \mathcal{D}_y - 1$

$$\varphi_p = \left[\frac{1}{\mathcal{D}_x^2 + \mathcal{D}_x\mathcal{D}_y + \mathcal{D}_y - 1} \right] \sin(x + 2y) = \left[\frac{1}{\mathcal{D}_y - 4} \right] \sin(x + 2y) = \left[\frac{\mathcal{D}_y + 4}{\mathcal{D}_y^2 - 16} \right] \sin(x + 2y)$$

$$= -\frac{1}{20}[\mathcal{D}_y + 4]\sin(x + 2y) = -\frac{1}{20}[2\cos(x + 2y) + 4\sin(x + 2y)]$$

such that the final solution $\varphi = \varphi_c + \varphi_p$ is

$$\varphi = e^{-x}\Phi_1(y) + e^x\Phi_2(y - x) - \frac{1}{20}[2\cos(x + 2y) + 4\sin(x + 2y)]$$

Example 8.5. Find the solution of this partial function, $[\mathcal{D}_x - \mathcal{D}_y + 2][\mathcal{D}_x + \mathcal{D}_y - 1]\varphi = x^2y$

1) Complementary φ_c : Compared to $\mathcal{D}_x - m\mathcal{D}_y - a$

$$[\mathcal{D}_x - \mathcal{D}_y + 2][\mathcal{D}_x + \mathcal{D}_y - 1] = 0 \longrightarrow m_1 = 1, a_1 = -2 \text{ and } m_2 = -1, a_2 = 1, \text{ such that}$$

$$\varphi_c = e^{-2x} \Phi_1(y+x) + e^x \Phi_2(y-x)$$

2) Particular φ_p : $[\mathcal{D}_x - \mathcal{D}_y + 2][\mathcal{D}_x + \mathcal{D}_y - 1] = \mathcal{D}_x^2 + \mathcal{D}_x - \mathcal{D}_y^2 + 3\mathcal{D}_y - 2$

$$\begin{aligned} \varphi_p &= \left[\frac{1}{\mathcal{D}_x^2 + \mathcal{D}_x - \mathcal{D}_y^2 + 3\mathcal{D}_y - 2} \right] x^2y = -\frac{1}{2} \left[\frac{1}{1 - \frac{1}{2} \underbrace{(\mathcal{D}_x^2 + \mathcal{D}_x - \mathcal{D}_y^2 + 3\mathcal{D}_y)}_u} \right] x^2y \\ &= -\frac{1}{2} \left[1 + \frac{u}{2} + \frac{u^2}{4} + \frac{u^3}{8} \right] x^2y = -\frac{1}{2}x^2y - \frac{3}{4}x^2 - \frac{1}{2}xy - \frac{3}{2}x - \frac{3}{4}y - \frac{21}{8} \end{aligned}$$

such that the final solution $\varphi = \varphi_c + \varphi_p$ is

$$\varphi = e^{-2x} \Phi_1(y+x) + e^x \Phi_2(y-x) - \frac{1}{2}x^2y - \frac{3}{4}x^2 - \frac{1}{2}xy - \frac{3}{2}x - \frac{3}{4}y - \frac{21}{8}$$

Example 8.6. Evaluate this partial differential function, $[\mathcal{D}_x - 3\mathcal{D}_y - 2]^2 \varphi = 2e^{2x} \sin(3x+y)!$

1) Complementary φ_c :

$$[\mathcal{D}_x - 3\mathcal{D}_y - 2]^2 = 0 \longrightarrow m = 3, a = 2, \text{ such that}$$

$$\varphi_c = [\Phi_1(y+3x) + x\Phi_2(y+3x)]e^x$$

2) Particular φ_p :

$$\begin{aligned} \varphi_p &= \left[\frac{1}{(\mathcal{D}_x - 3\mathcal{D}_y - 2)^2} \right] 2e^{2x} \sin(3x+y) = 2e^{2x} \left[\frac{1}{\mathcal{D}_x^2 - 6\mathcal{D}_x\mathcal{D}_y + 9\mathcal{D}_y^2} \right] \sin(3x+y) \\ &= 2e^{2x} \left[\frac{1}{-9 + 18 - 9} \right] \sin(3x+y) = 2e^{2x} \left[\frac{1}{0} \right] \sin(3x+y) \quad (\text{failed}) \\ &= 2e^{2x} \underbrace{\left[\frac{1}{\mathcal{D}_x - 3\mathcal{D}_y} \right] \left[\frac{1}{\mathcal{D}_x - 3\mathcal{D}_y} \right] \sin(3x+y)}_{\substack{A \\ B}} \end{aligned}$$

$$A = \int \sin(c) dx = x \sin(y+3x) \quad (a=3, y=c-3x \rightarrow \sin(c))$$

$$B = \int x \sin(c) dx = \frac{1}{2}x^2 \sin(y+3x) \quad (a=3, y=c-3x \rightarrow x \sin(c))$$

such that the final solution $\varphi = \varphi_c + \varphi_p$ is

$$\varphi = [\Phi_1(y+3x) + x\Phi_2(y+3x)]e^x + x^2e^{2x} \sin(y+3x)$$

Exercise 10. Solve the following differential equations!

$$1. \frac{\partial^2 \varphi}{\partial x^2} + 2 \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial y^2} + 2 \frac{\partial \varphi}{\partial y} + 2 \frac{\partial \varphi}{\partial y} + \varphi = 0$$

$$2. [\mathcal{D}_x^2 + \mathcal{D}_x\mathcal{D}_y + \mathcal{D}_x - 6\mathcal{D}_y^2 + 13\mathcal{D}_y - 6] \varphi = 0$$

$$3. \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y} = 0$$

$$4. [2\mathcal{D}_x^2 + \mathcal{D}_x\mathcal{D}_y - 4\mathcal{D}_x - \mathcal{D}_y^2 + 2\mathcal{D}_y] \varphi = e^{x-3y}$$

$$5. [\mathcal{D}_x^2 + \mathcal{D}_x\mathcal{D}_y - 6\mathcal{D}_y^2] \varphi = x^2 \sin(x+y)$$

9 Applications of Differential Equations

Example 9.1. Find the output $x(t)$ if input $y(t) = 10!$ From the figure, it can be said that

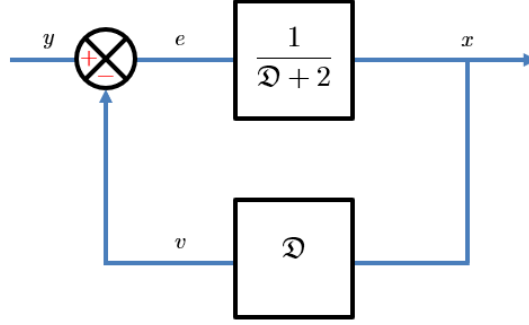


Figure 1: Block diagram system

$v = \mathcal{D}x$ and $e = y - v$, therefore

$$x = \left[\frac{1}{\mathcal{D} + 2} \right] (y - \mathcal{D}x) \longrightarrow \left[1 + \frac{\mathcal{D}}{\mathcal{D} + 2} \right] x = \left[\frac{\mathcal{D}}{\mathcal{D} + 2} \right] y$$

such that,

$$x = \left[\frac{1}{2\mathcal{D} + 2} \right] 10 = \left[\frac{1}{2} - \frac{\mathcal{D}}{2} \right] 10 = 5$$

Example 9.2. A mass $m = 1$ kg is placed as shown in Fig.(2a), having the damper $b = 2$ Ns/m and spring $k = 5$ N/m. A wall where it vibrates in the horizontal axis with $y(t) = 2 \cos(2t)$. Find the response of the system!

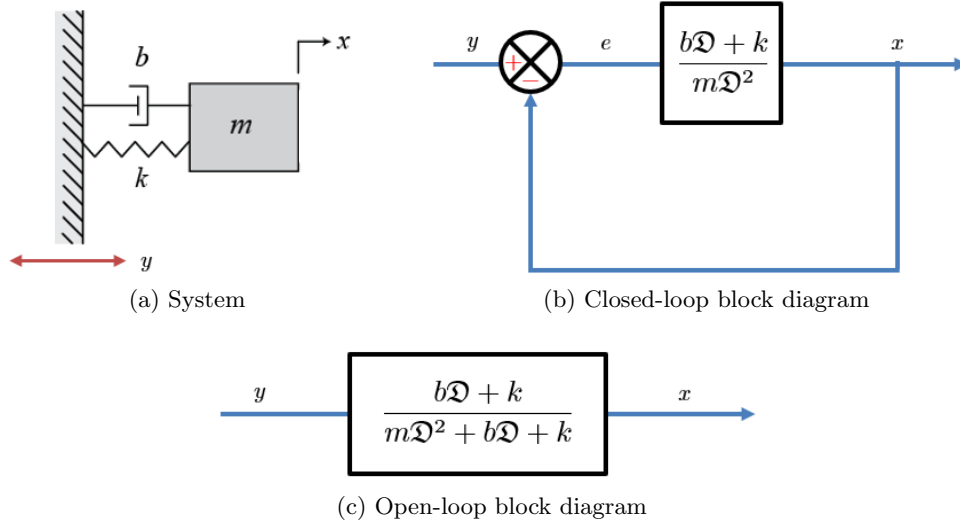


Figure 2: The system and its block diagram (closed- and open-loop)

$$m\ddot{x} + b(\dot{x} - \dot{y}) + k(x - y) = 0 \longrightarrow m\mathcal{D}^2 + b\mathcal{D}(x - y) + k(x - y) \longrightarrow m\mathcal{D}^2 x = (b\mathcal{D} + k)(y - x)$$

$$\frac{x}{y} = \frac{b\mathcal{D} + k}{m\mathcal{D}^2 + b\mathcal{D} + k}$$

From the equation, the complementary solution x_c is

$$[\mathcal{D}^2 + 2\mathcal{D} + 5] = 0 \longrightarrow \mathcal{D}_{1,2} = -1 \pm 2i, \text{ such that } x_c = c_1 e^{-1+2i} + c_2 e^{-1-2i}$$

The particular solution x_p is

$$\begin{aligned} x_p &= \left[\frac{2\mathcal{D} + 5}{\mathcal{D}^2 + 2\mathcal{D} + 5} \right] 2 \cos(2t) = 2 \left[\frac{2\mathcal{D} + 5}{2\mathcal{D} + 1} \right] 2 \cos(2t) = 2 \left[\frac{4\mathcal{D}^2 + 8\mathcal{D} - 5}{4\mathcal{D}^2 - 1} \right] 2 \cos(2t) \\ &= -\frac{2}{17} [8\mathcal{D} - 21] \cos(2t) = \frac{2}{17} [16 \sin(2t) + 21 \cos(2t)] \end{aligned}$$

such that the final solution $x = x_c + x_p$ is $c_1 e^{-1+2i} + c_2 e^{-1-2i} + \frac{2}{17} [16 \sin(2t) + 21 \cos(2t)]$

Example 9.3. The RC-circuit is arranged in series as depicted in Fig.()3, find the equation of current in terms of time (t)!

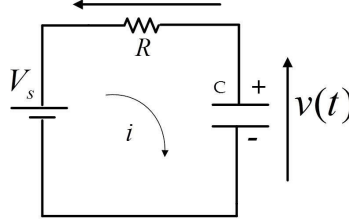


Figure 3: RC - circuit

$$Ri + \frac{1}{C} \int i dt = V_s \longrightarrow V_s = \text{constant}$$

The equation is then differentiated by time (t), such that

$$R \frac{di}{dt} + \frac{1}{C} i = 0 \longrightarrow (RC\mathcal{D} + 1)i = 0 \longrightarrow \mathcal{D} = -\frac{1}{RC}, \text{ such that } i(t) = Ae^{-\frac{1}{RC}t} + K$$

Example 9.4. The equations of two voltage in the two-loop circuit using resistance (R) and inductance (L) are presented as simultaneous differential equation,

$$\frac{di_1}{dt} + \frac{di_2}{dt} + 56i_1 + 40i_2 = 400 \quad \text{and} \quad \frac{di_2}{dt} - 8i_1 + 10i_2 = 0$$

Find the the equation of current in each loop (i_1, i_2) as a function of time (t) if $i_1(0) = 0, i_2(0) = 0$

$$(\mathcal{D} + 56)i_1 + (\mathcal{D} + 40)i_2 = 400 \quad \text{and} \quad -8i_1 + (\mathcal{D} + 10)i_2 = 0$$

To eliminate i_1 , the first equation is multiplied by 8 while the second $(\mathcal{D} + 56)$, therefore

$$\left. \begin{aligned} (\mathcal{D}x + 56) + (\mathcal{D}i_2 + 40) &= 400 & \xrightarrow{\times 8} & 8(\mathcal{D}x + 56) + 8(\mathcal{D}i_2 + 40) = 3200 \\ -8i_1 + (\mathcal{D} + 10)i_2 &= 0 & \xrightarrow{\times (\mathcal{D}+56)} & -8i_1(\mathcal{D}x + 56) + (\mathcal{D}x + 56)(\mathcal{D} + 10)i_2 = 0 \end{aligned} \right\} (+)$$

The complementary solution is

$$[\mathcal{D}^2 + 74\mathcal{D} + 880] = 0 \longrightarrow \mathcal{D}_1 = -15, \mathcal{D}_1 = -59, \text{ such that } i_{2c} = k_1 e^{-15t} + k_2 e^{-59t}$$

and the particular solution i_{2p} is

$$\left[\frac{1}{\mathcal{D}^2 + 74\mathcal{D} + 880} \right] 3200 = \left[\frac{1}{880} - \frac{74}{880^2} \mathcal{D} \right] 3200 = \frac{3200}{880} = 3.64$$

such that the final solution with respect to i_2 is $k_1 e^{-15t} + k_2 e^{-59t} + 3.64$ and with the same formula of elimination in terms of i_2 , it is obtained that the final solution of $i_1 = i_{1c} + i_{1p}$ constitute $c_1 e^{-15t} + c_2 e^{-59t} + 4.55$. Using the constraint given, it is found that the four parameters are $c_1 = 2.98, c_2 = -7.49, k_1 = -4.86$ and $k_2 = 1.22$, such that

$$\begin{aligned} i_1(t) &= 2.98e^{-15t} - 7.49e^{-59t} + 4.55 \\ i_2(t) &= -4.86e^{-15t} + 1.22e^{-59t} + 3.64 \end{aligned}$$

Exercise 11. Solve the following differential equations!

1. The block diagram of angular position control of servo-mechanic is presented as in Fig.(4) with input $\theta_1(t) = \beta t$ which is ramp function. Find the output response of the system along with the error in the setady state condition!

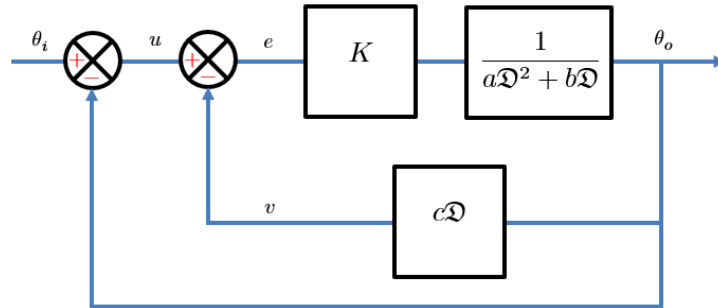


Figure 4: Block diagram of system

2. Given a differential function of system of cantilever with uniform loading, while one is bounded and another is free, is

$$EI \frac{d^2 y}{dx^2} = py - \frac{1}{2} wx^2 \longrightarrow y(0) = 0, y'(0) = 0$$

where p is external force, find the equation of deviation $y(x)$!

3. The RLC circuit is arranged in series as portrayed in Fig.(5) with $R = 160\Omega$, $L = 1$ H and $C = 10^{-4}$ F with the voltage of $V = 20$ V. Find the current and charge equation as a function of time (t)!

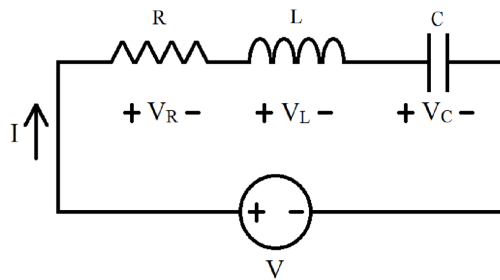


Figure 5: RLC circuit

10 Bank of Questions

This section would be released soon!